

Geostatistical Ore Reserve Estimation

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SYNOPSIS

Matheron's geostatistical method for the estimation of ore reserves has been developed to the point at which real-life problems may be handled effectively. The steps involved in the method are:

- (i) obtaining the variogram,
- (ii) fitting a model, and
- (iii) producing block estimates.

All three steps may cause difficulty, and would-be users need a deep understanding of the theory if failure is to be avoided.

To illustrate these points, specific cases are considered:

- (i) two almost equidimensional porphyry deposits, one drilled randomly and the other regularly; and
- (ii) two iron formations, one drilled extensively and being mined, while the other is still at the stage of early estimation.

INTRODUCTION

To many, the theory of regionalized variables as developed by Matheron (1966) is an elusive concept. Matheron's (1963) original paper in English is still relatively unknown, while Blais and Carlier's (1968) and David's (1969, 1970) attempt to popularize this theory and its practical applications appear to have had little impact in the English literature. Statisticians still find it difficult to grasp the geological inferences and repercussions of Matheron's theory (Watson, 1971).

In this context, this particular attempt towards standardization of geostatistical ore calculation may appear utopian to engineers concerned with grade control and ore reserve calculations. Yet, with the several papers written on the subject, a general approach is in the offing. What was said in previous years amounts more or less to the following four-point proposition: (i) compute the variogram of the deposit under study, (ii) fit the proper mathematical model, (iii) compute all the relevant variances, and (iv) do the kriging calculations.

Simple as it may seem, this general approach requires a good basic knowledge of the theory. Once variograms have been computed in one or more directions, it is often necessary to make certain simplifications in order to be able to detect the real mathematical structure of the phenomenon being studied, and to fit a model. Knowing what particular simplifications are permissible and which hazardous, a strict methodology can then be outlined and a package of standard computer programs can be designed for repetitive use.

STATEMENT OF THE PROBLEM

Our aim here is to outline the few theoretical notions that are essential to the solution of the following problem: the complete estimation of a very large deposit such as a porphyry copper or an iron formation, by means of accurate predictions of the true grade of individual mining blocks. In such cases, the information usually consists of 25 000 to 125 000 ft of diamond drilling and the samples are evenly cut. This latter requirement is essential.

We will first remark on the intrinsic hypothesis for regionalized variables and then discuss a number of problems arising in variogram computation, model fitting and kriging. Within this broad framework we will discuss the best arrangement for a series of computer programs and, importantly, we will refer to certain aspects which cannot be computerized and which are matters of intuitive judgment

or geological interpretation. Ore reserve estimation is herein treated as a 'manumatic' process (QuanSing, 1971), that is, both manual and automatic.

BASIC THEORY OF REGIONALIZED VARIABLES: THE INTRINSIC HYPOTHESIS

This theory considers grade variations in a deposit as one realization of an order-two, stationary-increment random function. This means that the grade at any two points separated by h is such that

$$E[f(x) - f(x+h)] = m(h)$$

$$E\{[f(x) - f(x+h)]^2\} = 2\gamma(h)$$

This model which is developed for point values bears consequences for the type of sample values applying to ore estimation, that is, samples characterized by a specific volume, shape and orientation. The problem of *size* of the samples must be faced and its bearing on the variogram $2\gamma(h)$ must be analyzed.

If the intrinsic hypothesis does not hold, that is, if drift is present, we become concerned with the intricate problems of optimum estimation in non-stationary sets (for a complete presentation of the theory, see Matheron, 1971).

Our original four-point proposition will now be analyzed and discussed, using as a basis for comparison and practical considerations four contrasting types of deposits. These are:

Deposit A: polymetallic porphyry deposit yielding three different metals and which has been drilled at almost regular intervals along sections.

Deposit B: huge low-grade porphyry copper deposit which has been drilled in every direction and at all possible angles due to unusually difficult topographic conditions.

Deposit C: flat-lying iron formation (taconite) with a regular grid of vertical diamond-drill holes.

Deposit D: meta-taconite rich in specular hematite and magnetite, folded into an overturned syncline, and drilled at fairly regular intervals along sections 400 ft apart over a strike length of about 8 000 ft.

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COMPUTATION OF THE VARIOGRAM FUNCTION

In order to perform any geostatistical calculation on a deposit we must first define its variogram function, that is, we must find a mathematical equation that represents fully the grade variations within it. This is not always easy, especially when the deposit is drilled irregularly or when the grade along strike differs from that normal to strike.

In deposit A, for example, we obtain different variograms for each of the 45 holes across it; the curves are not only different but they also yield different numerical values (a few of these curves are shown in Fig. 1).

In other instances, we may obtain nothing significant in terms of geostatistical structure, as in deposit B, for example. In this deposit, each of the 100 drill holes available yields a variogram such as those shown in Fig. 2. Another disheartening example for the beginners is the parabolically increasing variogram shown in Fig. 3a. In contrast, variograms of Fig. 3b and Fig. 4 present no problems.

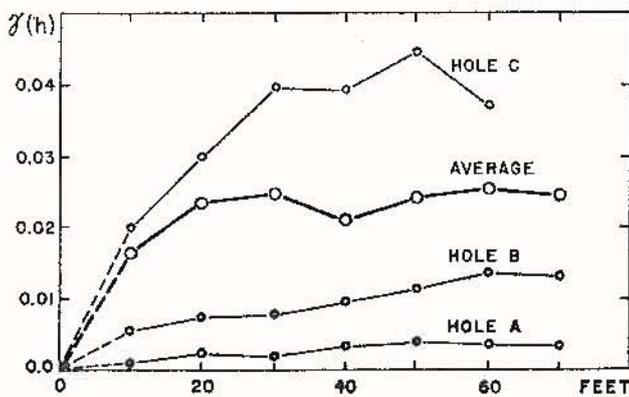


Fig. 1. Deposit A — metal No. 2. Sample variograms of drill holes, 10-ft samples.

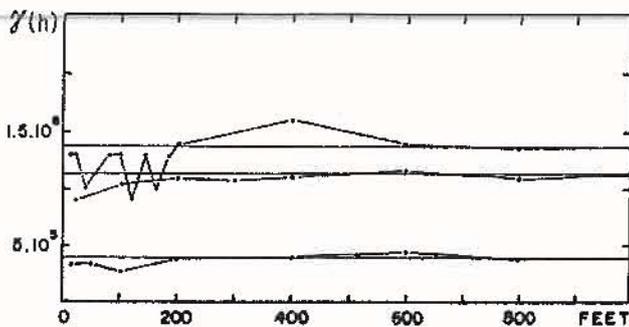


Fig. 2. Deposit B. Sample variograms of drill holes.

Bearing in mind the theory of regionalized variables, we show below how we have determined the intrinsic dispersion of values in each of these four deposits.

Deposit A

A first look at all the individual variograms from drill holes in this porphyry deposit reveals a consistent transition feature, but the plateau ("sill") and range values of these variograms are very variable, the plateau values especially.

In order to put some order into this, we plot the mean grade versus the standard deviation for each hole. As shown in Fig. 5, a fairly good linear fit is obtained, which allows the use of a multiplicative factor when dealing with the various variograms. This is linked to the so-called proportional effect, which states that dt/t is a constant. This favourite of geologists

is also responsible for the existence of the lognormal law and its related properties. Had we computed the variograms with logarithmic values, all curves would have been much more similar to each other.

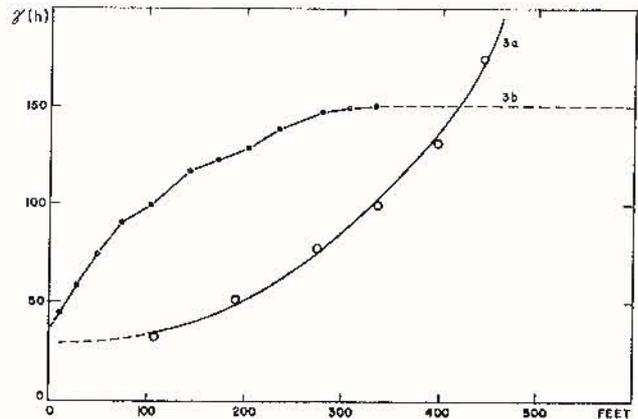


Fig. 3. Deposits C and D. Composite variograms — the good and the bad.

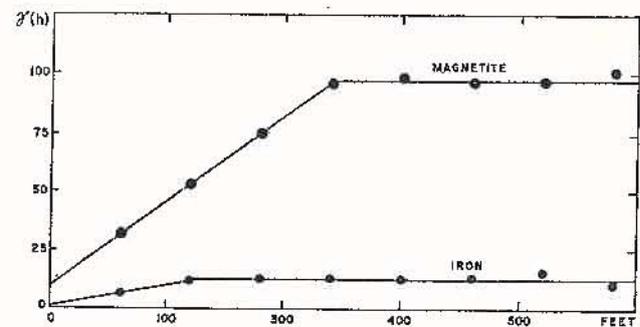


Fig. 4. Deposit D. Composite vertical variograms of 60-ft samples.

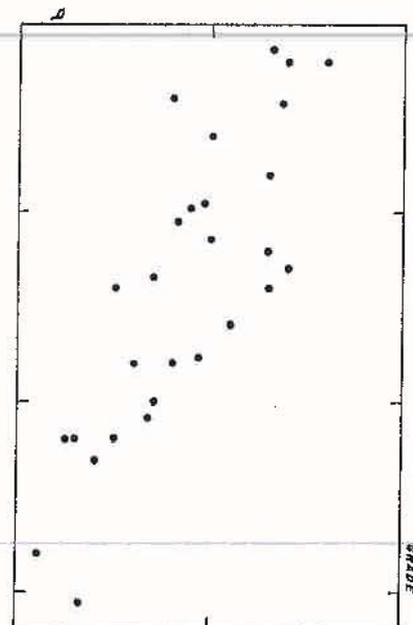


Fig. 5. Deposit A. Correlation between grade and standard deviation for each hole.

It should also be understood that variograms for individual holes may be very irregular, because our estimate of the local variogram is itself a random variable. Large variances are involved and these become increasingly large as h increases.

Thus, as a rule of thumb, we can safely disregard major fluctuations in the variogram curve when the variogram has been computed from less than 30 couples or when h is larger than one third of the total field length.

The major problem here is to depict unequivocally the grade variations within the entire deposit. This is done by averaging all the available variograms on parallel or sub-parallel drill holes. A variogram for each of two mutually perpendicular directions is obtained. These two variograms, being closely similar, are averaged to yield a unique smooth curve, which is the model sought (see Fig. 1). Had these two general variograms been appreciably different, we would have had to correct for anisotropy in one of the two directions. Anisotropy in the third direction will be checked with a different type of variogram, as explained below in the section dealing with deposit B. For the time being, however, we are satisfied with isotropy in two directions and relatively homogeneous geological conditions.

Deposit B

As explained previously, this porphyry deposit containing billions of tons of ore has been drilled in very irregular fashion. Most holes cross this equidimensional body completely and yield 100 or more evenly-cut 10-ft core samples. Although there are about 100 such holes, very few are parallel to each other.

To start with, we compute the variogram function for a number of individual holes. Some of the curves obtained are shown in Fig. 2. These exhibit an almost perfect total nugget effect. From a kriging standpoint, these variograms are useless because they indicate the absence of relationship between two adjacent 10-foot samples.

However, can we really extrapolate this result to 200-foot blocks? As this is the size of mining blocks in which we are interested, the question is not one of semantics. From a geological point of view, one would expect a relationship between the grades of these blocks. The individual 10-foot core samples may, in fact, be too large to show fine structures (the mineralization being in fractures) and too small to show broad variations.

The bigger the blocks are, the smaller their variance should become. If the block grades are structurally related, then their variance should decrease at a lower rate than the corresponding nugget effect. This effect is proportional to $1/\nu$ while the variance (plateau of variogram) decreases according to $(1 - F(\nu))$, where $F(\nu)$ is the average value of $\gamma(h)$ in volume ν .

So, we compute variograms on 50-foot sample increments. The result is somewhat better than with 10-foot samples, but it is not until we use 200-foot samples that decisive progress is made. However, since the number of parallel 200-foot samples is far too limited, we must tackle the problem differently, as explained below.

We use the *cluster technique* (David, 1970) and we divide the whole deposit into 200-foot cubical blocks. Each of these 1 400 blocks is intersected at various angles by one or more drill holes. The cluster of samples within each block can be regarded as a random sampling of this block. We take this cluster as our sample. As the block system is cubic, it is very easy to obtain a truly three-dimensional variogram since the variogram can be computed for any direction within the cubic lattice. This total variogram can be mapped only in four-dimensional space; however, we will satisfy ourselves with looking at sections of this hypersurface and thus obtain an important tool for checking possible anisotropies and for detecting rapidly the major axis of anisotropy if present.

Any geometric anisotropy is corrected easily by a simple affinity. When isotropy is restored, all the available variograms

can be averaged easily to yield a total variogram which describes unequivocally the entire deposit. The total variogram obtained in this case is shown in Fig. 6. This is far more reassuring than those of individual drill holes.

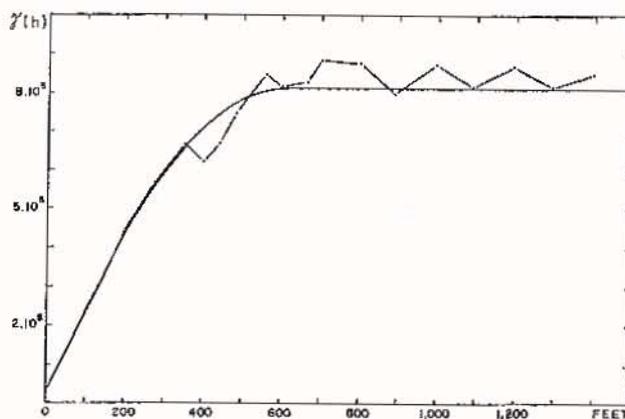


Fig. 6. Deposit B. Average variogram of clusters and fitted model.

With the above steps in mind, a series of computer programs can be visualized to study a large deposit systematically, for example:

- (i) to obtain a variogram for each hole and an average variogram for all parallel holes,
- (ii) to divide the entire deposit into cubical blocks of a given size,
- (iii) to compute the variogram for these blocks and to map sections of it, and
- (iv) to obtain an average 'isotropic' variogram.

It is not advisable to link all these routines into a major program for the simple manumatic reason that checks, interpretations and decisions need to be made at each step.

This same series of programs can be used in the case of regular drilling. In this case the clusters are reduced to parallel samples and a three-dimensional variogram descriptive of the whole deposit is obtained at once. This is the additional technique that was applied to deposit A and which allowed the check for anisotropy in the third perpendicular direction.

We have now described two difficult cases. The remaining two pertain to iron formations. One of them yielded almost perfect simple variograms, while the other took us into the complex field of universal kriging.

Deposit C

This flat-lying iron formation containing billions of tons of good-grade magnetic taconite has been investigated by means of a fairly regular grid of vertical diamond drill holes, with samples cut at regular five-foot intervals. The magnetic taconite portion is several hundred feet thick.

This simple geological situation presents a difficult ore estimation problem, summarized by the shape of variograms of individual drill holes, which is parabolic (Fig. 3b). This disquieting feature denotes an underlying trend which must be evaluated fully before any further treatment of the data. This type of problem arises not infrequently when one is dealing with bauxite or phosphate deposits where the geometry of all megascopic phenomena is very simple, but where an existing vertical trend in values masks the horizontal relationship from one hole to the next, although, geologically speaking, the general relations are obvious (Serra, 1970).

Problems of this kind lead us into the difficult world of universal kriging. In the following paragraphs, we will attempt to show why it is difficult and why this is no place for solitary attempts. A complete presentation of universal kriging is given in Matheron (1970).

A look at universal kriging. If a trend (we would rather say a 'drift') is present, the actual grade can be divided into a functional value plus a residual. Notice that we will never know what this functional form is; consequently, we will never know the true residuals but only estimates of them. Furthermore, to obtain an optimum estimate of the residuals requires knowledge of their variogram. Our sole avenue is to proceed by making certain assumptions and by checking them. Henceforth, we must assume

- (i) a variogram for the residuals (we call it the 'underlying' variogram),
- (ii) a type of drift, and
- (iii) a neighbourhood where this drift is valid.

From this, we deduce an estimate of the drift and, by difference, of the residuals. From these estimates we compute the variogram of the residuals which is compared with the expected variogram of residuals when the above three assumptions are simultaneously true, and when the residuals have been estimated optimally. This presents great bias problems since the shape of the experimental variogram tells us nothing about the underlying variogram of the real phenomenon being studied. For instance, the expectation of a linear variogram can appear as a parabolic experimental variogram. Complete familiarity with the theory is thus necessary before attempts to solve this kind of problem can be made.

Our series of programs in this case is simply the computation of the algorithm detailed by Matheron (1970) for a limited number of frequently valid sets of assumptions.

Problem of the 'practically regular' grid. This problem intervenes when attempts are made to compute the horizontal variograms from the estimated drift, to define the grade relationships along bedding. The 'practically regular' drilling grid on this deposit is in fact a 'random stratified' sampling: in each square of the grid superimposed on the entire deposit it is possible to find one drill hole which is in random position within the square. Under these conditions, how to obtain a variogram in a given horizontal direction and for a given lag?

To solve this problem, we have to compute all the possible couples of γ_H and group them in classes of distance ($h + \Delta h$) and angle ($\theta + \Delta\theta$). This operation is nothing more than a regularization of the data, the effect of which is shown in the next section. Note, however, that it is very simple to write a program to perform this algorithm and to detect any anisotropy in grade distribution. This technique has been described and used extensively by Marechal (1970). Theoretically, it would be simple to generalize it to a three-dimensional framework, but the number of mathematical operations then rapidly becomes prohibitive. One way to circumvent this difficulty is to use the cluster technique (David, 1970).

Deposit D

This is an orebody of several hundred million tons of meta-taconite, rich in specular hematite and magnetite, and folded into an overturned syncline reaching depths of 1 000 ft below the surface. About one third of the deposit has now been mined open cast and a wealth of sampling information is available from blast holes, bank samples and mill heads.

The orebody has been estimated from a few hundred vertical diamond drill holes located on a fairly regular 400 × 200-ft grid, yielding several thousand 10-foot core samples. Long- and medium-range planning of this mine are complicated by the fact that the ore is crudely but not evenly

stratified, with magnetite-rich layers in certain folded horizons and troublesome concentrations of anthophyllite in the trough of some synclinal dragfolds and along faults, as well as the need to appraise simultaneously variations in iron oxides, magnetic iron and grain size of both specular hematite and magnetite. On the other hand, daily or weekly ore grading is complicated by the difficulties of estimating accurately the grade of individual blasts and correlating closely the average grade of the 80 000 to 125 000 tons produced daily from four to eight contrasting digging points, with the average grade of mill heads.

As may be expected in such a complex geological situation, the variograms obtained from 10-foot samples for individual drill holes displayed large variations. Although several variograms were typical of pure random distribution, most curves showed a definite transitive feature, with the average variogram being an almost perfect curve (Fig. 3a).

The methodology was improved by doing the calculations on 60-foot sample increments, rather than using the original 10-foot core lengths. The length of these increments was chosen as 60 ft to conform to standard bench height, and thus to allow the calculation of average grade (iron and magnetite) of individual blocks 60 ft high and either 200 × 200 or 200 × 400 ft horizontally.

The average variograms obtained for the vertical direction are shown in Fig. 4. They are almost perfect examples of the transitive model. Significantly, iron and magnetite contents behave very differently. These exact numerical values represent no less than a formalization of the geological concepts used during development and mining of this orebody, such as those applied intuitively to laying out exploration drill holes, sampling of mining faces, and interpreting and calculating ore reserves.

With different transitive variograms obtained horizontally, that is, parallel to strike and normal to strike, it has been possible to calculate the essential geostatistical elements of the deposit in three-dimensional space.

FITTING THE MATHEMATICAL MODEL TO AN EXPERIMENTAL VARIOGRAM

Firstly, let us remark about the *size* of samples. Strictly speaking, we do not obtain *point* variograms since the samples are nearly always *segments* of core of length l . We need to be concerned about the relationship between the variogram for such core segments and the underlying model for point values. We must also concern ourselves with this relationship when samples are grouped into certain classes or increments. To assist in understanding these regularization problems, the following lines of mathematical formalism seem to be in order.

Regularization processes

If the grade at a given point is $f(x)$, then the grade of a sample of volume V centred on x will be

$$f_V(x) = \frac{1}{V} \int_V f(x+y) dy$$

Then, if we define as $k(y)$ the function which is such that $k(y) = 1$ when y belongs to V , and $k(y) = 0$ when y is outside V , we can write the grade of V as

$$f_V(x) = \frac{1}{V} \int f(x+y) k(y) dy$$

This is a convolution product of f and $k(-y)$, which we note as $\check{k}(y)$ or, in short,

$$f_V(x) = \frac{1}{V} f * \check{k}$$

We also have to introduce the geometric covariogram of V , which is defined as:

$$P = k * \overset{v}{k} = \text{Mes}(V \cap V-h).$$

The covariogram of the regularized variable may then be readily expressed as

$$K_V(h) = \frac{1}{V^2} f * \overset{v}{k} * \overset{v}{f} * k = \frac{1}{V^2} K * P$$

where K is the point covariogram.

Since $\gamma(h) = K(0) - K(h)$, we obtain

$$\gamma_V(h) = \gamma * P - \int \gamma(z) P(z) dz$$

It is also possible to obtain this regularized variogram by taking note that the half-variogram of samples of size V is equal to the estimation variance of a block V_i by another block V_j at distance h .

The above explains the relation between the *point* variogram and the *segment* variogram. In the second type of regularization, where samples are grouped into classes, the average variogram obtained is not the variogram corresponding to the average distance and angle for the simple reason that a variogram is not a linear operator. The exact operation is in two dimensions

$$\gamma_{\theta}^*(h) = \frac{1}{\Delta l \Delta \theta} \int_{-\frac{\Delta l}{2}}^{+\frac{\Delta l}{2}} \int_{-\frac{\Delta \theta}{2}}^{+\frac{\Delta \theta}{2}} \gamma(h+x, \theta+\varphi) dx d\varphi$$

It should be noted that the computation of a variogram based on the average value of the grades in each cell (block), such as carried out in the cluster technique, is a regularization process that differs from the above.

Choice of a point model

At this stage, we must determine exactly what we want to do with our information. Theoretically, the unique path would be to obtain a point variogram. However, this may involve deconvolution, which is not always a simple task. Two methods of simplification are suggested below.

The first is to work with metal weights (accumulations), with each drill hole becoming a point. Note that working in a space of clusters can produce the same effect (David, 1970).

The other method applies to samples that are small enough to be considered as mere points in relation to the dimensions of the orebody being studied. In order to know whether these simplifications are possible, it suffices to compute a few times the expression $\gamma * P$ for given geometrical figures and variograms.

A few well-known results concern the De Wisjjan variogram. For example, for a series of samples of length l and distant of h , if the point variogram is $\gamma(h) = 3\alpha \ln h$, its corresponding segment variogram is

$$\gamma(h) = 3\alpha \left(\ln \frac{h}{l} + \frac{3}{2} \right) \text{ for } h > 2l.$$

Another example of related computation is given by Marechal (1970) concerning the problem of irregularly-distributed samples; this author has shown the effect of this regularization on the anisotropy ratio and the range value when the point variogram answers to the spherical model. The following gives some idea of what is involved.

If there is geometric anisotropy, with a and b being the major axes, the estimates a^* and b^* are related to axes a and b as follows:

$$\frac{b^*}{a^*} + 1 = \frac{\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \alpha \sin^2 u} du}{\int_0^{\frac{\pi}{4}} \sqrt{1 - \sin^2 \alpha \sin^2 u} du}$$

and

$$b = \frac{4b^*}{n} \int_0^{\frac{\pi}{4}} \sqrt{1 - \sin^2 \alpha \sin^2 u} du$$

where

$$\cos \alpha = \frac{b}{a}$$

To sum up this section, point two of our original proposition, a suitable mathematical model must be found for the experimental variogram: De Wisjjan, linear, spherical, exponential, or a regularization of one of these models.

As a word of caution, usual least-squares estimates should not be used to fit a model, firstly, because the estimation variance of the variogram increases with h and, secondly, because we can use additional information, such as the variance of samples in the entire deposit. This variance is equal to $F(a, b, c) + C_0$ where, again, $F(a, b, c)$ is the average value of the variogram for a block with dimensions a, b, c and C_0 is the nugget effect. The determination of the nugget effect can also be made more precise by obtaining additional information such as duplicate drill holes and smaller samples (Marechal, 1970).

In all the cases we have studied, we have been able to fit spherical schemes to our experimental variogram curves. Having found a suitable model and having obtained a representative equation of the variogram, we can now finally solve our block-by-block estimation problem.

SEVERAL KRIGING METHODOLOGIES

Kriging has been described in several papers. The impression may have been gained that once the variogram is obtained, it is a straightforward process, especially when a large computer is available. However, to use this powerful mathematical tool properly and economically, it is important to keep in mind certain key principles.

To begin with, we say 'once the variogram is obtained . . .' In fact, the choice of our points in space depends very much on the problems involved in kriging. When using the cluster technique, for instance, we consider a cluster as a point because we assume that the grade of a block is the grade of the cluster of samples within it, to which is added a random error of zero mean; this gives us an easy way for solving the kriging of large blocks using only point kriging. This really is a co-estimation rather than an estimation. A short theoretical treatment of kriging is now required in order to show where the problems lie.

It should be remembered that a kriged estimate Z^* of the grade of a given block is the linear combination of available sample grades X_i which satisfies the following conditions:

- (i) the estimate must be unbiased, whatever the true unknown mean grade of the deposit may be, and
- (ii) it has a minimum estimation variance.

This leads to the usual system of linear equations, one system for each block i_0 which needs solving. In this system, λ_i is the unknown weighting coefficient, μ is a Lagrange

multiplier, also unknown, σ_{ij} is the covariance of elements i and j and we have

$$\begin{cases} \sum_i \lambda_i \sigma_{ij} = \sigma_{i_0 j} + \mu \\ \sum_i \lambda_i = 1 \end{cases}$$

Thus, from a computation point of view, two limitations can occur: the system should not be too large and the covariances should be easy to compute.

The large grid case

We dispose of these two limitations when the variogram displays a plateau (sill) and when all the covariances are equal to zero, except for the covariance of the samples inside the block in relation to the block itself. This is called 'large-grid' kriging on account of the size of the block compared with the range of the variogram. As shown by Matheron (1971), all that matters here is to count the number of samples within the block and those outside. Providing that the smallest distance between two samples is larger than $2a$, where a is the range of the transitive variogram, we obtain in two-dimensional space:

(i) all the n inner samples have the same weight:

$$\lambda = \frac{1}{n + n'} + \frac{n}{n + n'} \frac{A}{CS},$$

(ii) all the n' outer samples have the same weight:

$$\lambda' = \frac{1}{n + n'} - \frac{n'}{n + n} \frac{A}{CS},$$

where S is the surface of the block, A is the average value of the covariogram

$$A = 2 \int_0^{\infty} zK(z) dz,$$

and C is the plateau value of the variogram.

This, however, is not the usual situation. We encountered it at the early stages of sampling deposit C when the drilling grid was very loose. This method can nevertheless be used to some advantage for selecting target areas when it is not advisable or economical to test-drill at closer spacings. It also has the advantage of being workable with a slide rule only.

Reducing the size of the system

The number of equations to solve is equal to one plus the number of samples which must be taken into account to estimate a given block of ore. Elementary calculations and common sense teach us that a *screen effect* is to be expected.

If we have a series of samples surrounding a block, there is very little point in considering other samples further away unless the variogram shows a high nugget effect. Practically speaking, this means that in a cubic block system each individual block will need to be estimated from the samples of the 26 blocks surrounding it, plus itself. In several instances we have encountered as many as 200 samples in these peripheral blocks. Usually, the blocks are of the order of 100 ft, and the sample increments used are 10 ft in length. We gain no new information by considering each sample increment individually and it is generally preferable to consider the complete intersection of a hole in a given block. By gaining here on the number of samples, we lose there on the computation of the covariances. Another aspect which is theoretically important but which is rarely encountered in practice, is the symmetry of distribution of samples around a

given block. All equidistant samples bring the same contribution to kriging, so that they can be replaced by their average value.

Covariance computations

What is exactly the covariance of a sample and a block? If we look back at the definition and do not wish to avoid integrals, we find that the covariance $Cov(v, V)$ of a sample v and a block V is nothing more than a constant minus the average value of the variogram (a vector function) when one end of vector h describes v and the other independently describes V , or as shown in Fig. 7:

$$Cov(v, V) = C - \frac{1}{vV} \int_v \int_V \gamma(x-y) dx dy$$

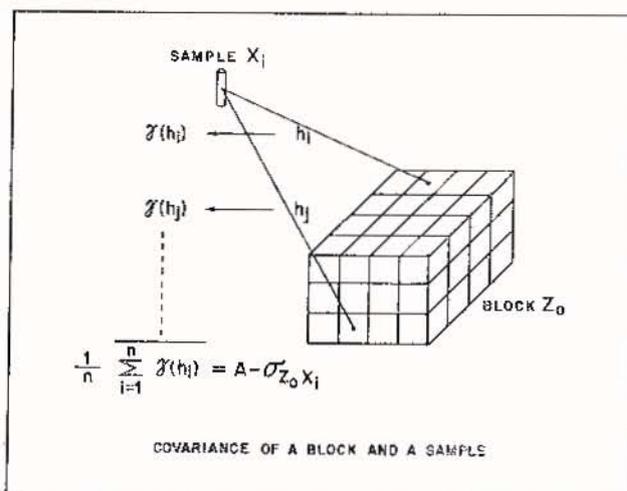


Fig. 7. Computation of the covariance of a block and a sample.

With the exception of the sphere, no block is simple enough to yield an analytical form allowing easy computation of the covariance. Hence, we must seek approximations. A major simplification occurs when the two volumes are sufficiently far apart that their distance is large compared with their dimensions, so that their covariance is equal to the covariance of their centers of gravity. If not, a valid approximation is obtained by estimating the integral through a discrete summation. Here again, no sophisticated triple integration algorithm is necessary. The convergence of $Cov(v, V)$ is usually obtained quickly, that is, without dividing the block into an excessively large number of elements. This makes it very simple to program the covariance of anything with anything else. However, this summation process can be time-absorbing on the computer.

For additional simplification, it is interesting to look at the case when the samples are located randomly in each block. Following the theory of random kriging, we find that instead of recomputing each time the exact covariance of a particular sample and a given block, this covariance is nothing more than the expected value of the covariance of a block and a sample located randomly in the other block. This is the exact definition of the covariance of two blocks (Serra and Marechal, 1970).

If we can use a model such as that of deposit B (Fig. 6), then things are far more simple. As mentioned previously, in this model we have in each block a cluster of samples, which allows us to assume that the grade of the block is the average grade of the cluster plus a random error. The only covariances intervening are those of clusters i and j . Since the experimental variogram is that of the clusters, no integration is necessary.

This is very similar to point-kriging, which appears to be the most important practical method.

Finally, when the relative size of the problem and the capacity of the computer are such that it is possible to estimate very small unit blocks by ascribing to them the grade of their centers, the optimum grade of any block can be reconstructed by summing up the grades of the unit blocks, and no integration whatever is required. This technique has the additional advantage of lending itself readily to the preparation of grade contour maps.

Programming and kriging

Here again it is a perceptible illusion to hope for a computer program adapted to all kriging cases. Once the type of problem to be solved is recognized fully, a two- or three-dimensional computer program best adapted to the situation is selected according to how it is decided to treat the covariances. These programs produce maps of block grades level by level, or section by section, for instance. It is a simple matter to produce these maps at a given scale and to include in each block all relevant information, including the kriging variance which indicates how accurate the estimated grade of a particular block actually is.

The kriging variance is not computed when the estimated grade is to be shown as contour maps. In this case, we do point-kriging with the smallest economical grid. We thus obtain the best possible map, but then it is more difficult to give precisely the estimation variance of a planimetered block, for instance. This problem also appears when the precision of the kriged grade estimates of a system of regular blocks is required.

RECONCILIATION OF GEOSTATISTICS, GEOLOGY AND MINING NECESSITIES

Geological considerations must guide the entire ore appraisal. In the four cases we have discussed, the ore structures vary enormously. In the porphyry coppers studied (deposits A and B) the structural controls are extraordinarily complicated, with billions of mineralized fractures, whereas in the flat iron formation (deposit C) the structure is very simple, and in the folded and metamorphosed iron formation (deposit D) it is more complex. Yet, the same basic mathematical approach is applicable to all four deposits.

It should be stressed, however, that with a particular 'geological direction', the properties of the ore are relatively constant along strike or along dip, and generally very variable across the structures, such as a line of samples crossing a fold structure. Variograms should never be computed on such a line.

Secondly, the geologist's advice is particularly important when ore boundary problems are dealt with and automatic kriging used. Unless a digitizer is available, it is very difficult to define appropriately the limits of ore in a kriging program. For example, the automatic program may well have provided a grade for a block of rock which is deep in the footwall or the hangingwall. A good practice is to superimpose on the map of kriged grade a transparent overlay of the geological interpretation, and then to trace the ore boundaries manually. It should be noted that the calculated boundary may not be too precise because the kriged grades are in themselves approximations only, each having its own variance.

Once grade contours are defined, it is a simple matter to compute the grade-tonnage curve by drawing the cumulative distribution of the estimated block grades. This curve will not be the true curve for the deposit, but whether a block is mined or not, or falls into category A or B, depends upon the estimates that can be made and not on the real grades, which exist but remain unknown until the ore is mined out and milled. Thus, as knowledge of a deposit improves, so does the grade-tonnage curve.

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REFERENCES

- BLAIS, R. A., and CARLIER, P. A. (1968). Application of geostatistics in ore valuation. *Ore Reserve Estimation and Grade Control*. C.I.M.M. Special Vol. 9, pp. 41-68, Montreal.
- BUBENICEK, L., and HAAS, A. (1969). Method of calculation of the iron ore reserve in the Lorraine deposit. *A Decade of Digital Computing in the Mineral Industry*. A.I.M.E. Special Vol. pp. 179-210. New York.
- DAVID, M. (1969). The notion of extension variance and its application to the grade estimation of stratiform deposits. *A Decade of Digital Computing in the Mineral Industry*. A.I.M.E. Special Vol. pp. 63-81. New York.
- DAVID, M. (1970). Geostatistical ore reserve calculation, a step by step case study. *IXth International Symposium for Decision-Making in the Mineral Industry*. C.I.M.M. Special Vol. no. 12. Montreal.
- DAVID, M. (1970). The geostatistical estimation of porphyry-type deposits and scale factor problems. *Proceedings, Příbram Mining Symposium*. Příbram, Czechoslovakia.
- HUBBREGIS, CH., and MATHERON, G. (1970). Universal kriging—an optimal approach to trend surface analysis. *Decision-Making in the Mineral Industry*. C.I.M.M. Special Vol. no. 12. Montreal.
- KRIGE, D. G., et al. (1970). The use of contour surfaces as predictive models for ore values. *A Decade of Digital Computing in the Mineral Industry*. A.I.M.E. Special Vol. pp. 127-161.
- MARECHAL, A., and ROULLIER, J. P. (1970). Etude géostatistique des gisements de bauxite. *Revue de l'Industrie Minière*, vol. 52, no. 7, pp. 492-507.
- MARECHAL, A., and SERRA, J. (1971). Random kriging. *Geostatistics*, D. F. Merriam Editor, Plenum Press, New York.
- MATHERON, G. (1963). Principles of geostatistics. *Economic Geology*, vol. 58, pp. 1246-1266.
- MATHERON, G. (1965). *Les variables régionalisées et leur estimation*, Masson et Cie, editor, Paris, 300 pp.
- MATHERON, G. (1967). Kriging or polynomial interpolation procedures. *C.I.M. Bulletin*, vol. 60, no. 655, pp. 1041-1045.
- MATHERON, G. (1968). *Osnovy Prikladnoi geostatistiki*. Mir, Moscow, 300 pp.
- MATHERON, G. (1971). Théorie des variables régionalisées. *Cahiers du Centre de Morphologie Mathématique*, Fontainebleau.
- QUAN-SING, G. (1971). Private communication.
- SERRA, J. (1967). *Echantillonnage et estimation des minerais de fer Lorraine*. Doctorate Thesis, 690 pp. Nancy.
- SERRA, J. (1968). Les structures gigognes, Morphologie Mathématique et interprétation métallogénique. *Mineralium Deposita*, vol. 3, pp. 135-154.
- SERRA, J. (1970). Quantification of lateritic type structures and application to their estimation. *Decision-Making in the Mineral Industry*. C.I.M.M. Special Vol. no. 12. Montreal.
- WATSON, G. S. (1971). Trend surface analysis. *Journal of International Association for Mathematical Geology*, vol. 3, no. 3, pp. 215-226.

