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Deformation of stratified rock masses: A laminated model

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SYNOPSIS

In view of an urgent need for a simple means of simulating the behaviour of stratified rock masses, a long-dormant linear model is revived, generalized, and discussed in some depth. The model is piece-wise homogeneous, consisting of a pile of homogeneous isotropic laminae in which the interfaces between beds are parallel and free of shear stresses and cohesion. The continuity of stresses and normal displacements is assured across the bedding planes and, to maintain simplicity, the theory of thin plates is used as the kernel of the model.

The paper covers the case where the rock mass consists of discrete beds in which the layers are arranged either heterogeneously or systematically. The stratification is regarded as *uniform* when all the layers are of the same thickness and their Young's modulus changes exponentially with depth; and as *homogeneous* when all the laminae have identical thickness and properties. The case of uniform stratification is also treated on the basis of an equivalent quasi-continuum derived from the basic model. Attention is focused on the derivation of influence functions. It is demonstrated that the quasi-continuous models, when they are available, can replace their distinct-layer counterparts.

The homogeneous quasi-continuum, which permits the derivation of its influence functions in closed form, facilitates the determination of the roof and floor convergence. In the case of a significant class of problems, the convergence distribution is defined by a second-order differential equation in the *xy* plane, which simplifies to a Poisson's equation if the mining excavations are unsupported. With this result, the paper prepares for the solution of a host of practical problems. These solutions are presented in a series of papers that are in the course of publication.

SAMEVATTING

Met die oog op 'n dringende behoefte aan 'n eenvoudige manier om die gedrag van gelaagde rotsmassas na te boots, word 'n lineêre model wat lank in onbruik was, weer bygehaal, veralgemeen en taamlik indringend bespreek. Die model is stuksgewyse homogeen en bestaan uit 'n stapel homogene, isotropiese lamelle waarin die skeidingsvlakke tussen die lae parallel en vry van skuifspannings en kohesie is. Die kontinuïteit van spannings en normale verplasings word oor die laagvlakke verseker en met die oog op eenvoud word die teorie van dun plate as die kern van die model gebruik.

Die referaat dek die geval waar die rotsmassa bestaan uit afsonderlike strata waarin die lae heterogeen of sistematies gerangskik is. Die stratifikasie word as eenvormig beskou wanneer al die lae ewe dik is en hul Youngmodulus eksponensiaal met die diepte verander; en as homogeen wanneer al die lamelle dieselfde dikte en eienskappe het. Die geval van eenvormige stratifikasie word ook behandel op die basis van 'n ekwivalente kwasiekontinuum wat van die basiese model afgelei is. Die aandag word toegespits op die afleiding van invloedfunksies. Daar word getoon dat die kwasiekontinue modelle, waar hulle beskikbaar is, hul afsonderlike laagteenvoeters kan vervang.

Die homogene kwasiekontinuum wat die afleiding van sy invloedfunksies in geslote vorm moontlik maak, vergemaklik die bepaling van die dak- en vloerkonvergensie. In die geval van belangrike klasprobleme word die konvergensieverdeling omskryf deur 'n tweedeorde-differensiaalvergelyking in die *xy*-vlak, wat tot 'n Poissonvergelyking vereenvoudig word as die mynuitgrawings ongestut is. Met hierdie resultaat baan die referaat die weg vir die oplossing van 'n menigte praktiese probleme. Hierdie oplossings word aangebied in 'n reeks referate wat tans vir publikasie voorberei word.

Introduction

During the 1950s an intensive programme of research was initiated by the late Professor E.L.J. Potts at the Department of Mining Engineering, King's College, University of Durham (now University of Newcastle upon Tyne). Much of this effort was focused on the behaviour of stratified sedimentary rock masses, which are usually associated with coal mining.

As part of this general effort, a paper was published in 1961 entitled 'An introductory mathematical analysis of the movements and stresses induced by mining in stratified rocks'[1]. An approximate linear model purporting to simulate the behaviour of horizontally stratified rock masses was introduced in the paper. This model, which was later termed the 'frictionless laminated model'[2], is based on a mathematical abstraction of the behaviour of a pile of horizontal thin plates in which the interfaces between the plates are frictionless and, consequently, free to slide.

The model, partly because of a shift in interest on the part of its originator and partly, perhaps, because of the perturbing simplifications involved in its construction, largely fell into disuse after the mid-1960s. During the last few years, however, the author's interest in the behaviour of laminated rock masses has been re-awakened. A study of recent reports[3,4] on the prediction of surface subsidence caused by coal mining appears to suggest that the frictionless laminated model is the simplest, and may even be the best, available predictor of surface disturbances induced by underground coal extraction.

This observation provides a powerful motivation for the exploration of the capabilities of this model. This paper represents the first step in this direction.

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Historical Review

Parallel to the work at Newcastle upon Tyne, an intensive search began, some three decades ago at the Mining Engineering Department, University of Nottingham, for a suitable linear model to describe the behaviour of strata associated with coal seams. The pioneering work of Hackett[5,6], Berry[7], and Berry and Sales[8,9] was based on either the homogeneous isotropic elastic model or the homogeneous transversely isotropic elastic model, both of which fulfil all the requirements of classical continuum mechanics. The research was motivated largely by the need to predict the subsidence of the ground surface resulting from underground mining of coal.

The rock mass surrounding coal seams usually consists of distinct layers. An individual stratum may possess physical properties and a thickness that are different from those of other beds. Observations suggest that some of these beds undergo movements relative to one another at the interfaces. It cannot be a source of surprise, therefore, that the homogeneous isotropic model does not in most instances provide an acceptable description of the distortion of the ground surface, regardless of the numerical values chosen for the Young's modulus and the Poisson's ratio[7]. The model appears to lack the pliability exhibited by the stratified overburden encountered in coal fields. It predicts subsidence which, in comparison with field observations, spreads unrealistically widely and is too small in magnitude over the central region of the extracted area.

At least three serious attempts appear to have been made to simulate the deformation of coal measures. First, after their disappointing experience with the isotropic model, Berry and Sales suggested that a homogeneous transversely isotropic medium is more likely to yield realistic predictions[8,9]. In fact, it was shown later[10] that, in certain circumstances, a transversely isotropic substance can be regarded as being 'equivalent' to a stratified continuum. Unquestionably, this model has considerable appeal and should not be discounted. However, it has not gained universal acceptance, perhaps largely because considerable conceptual and practical difficulties hinder the determination of the numerical values of its five elastic constants.

Chronologically, the next attempt involved the introduction of a simple stratified model, which permits free sliding at the interfaces between laminae. Thus, to achieve greater flexibility, some of the constraints that are usually imposed on classical models by the requirements of continuity of displacements and stresses are relaxed. The simplest version[1] of this frictionless laminated model, which was mentioned in the Introduction, was described in 1961. Much of the remainder of this paper is devoted to the derivation of equations controlling the general version of this model.

Much more recently, Wardle called attention to the practicability of solving problems in a multi-layered medium[11]. Using ingenious mathematical and powerful computational techniques, he developed a numerical method to assist in the solution of ground-control problems related to stratified rock masses[12]. Wardle's method can handle either a laminated continuum or a mass with frictionless contact surfaces between its neighbouring beds. In both cases, each layer is homogeneous, isotropic or transversely isotropic, with individually specified moduli and thicknesses.

In principle, Wardle's approach is sufficiently general to incorporate the older frictionless laminated model. However, its very generality prevents it from revealing some of the interesting and powerful features of the less general formulation.

Scope of the Paper

Research during the past two years, utilizing work done intermittently during the past two-and-a-half decades, has shown that the frictionless laminated model appears to have the potential to simulate reasonably well the deformation of strata above and below coal seams. Also, a simplified version of the model has mathematical features that facilitate the convenient solution of practical problems such as the computation of convergence, the prediction of surface subsidence, the estimation of pillar loads, the analysis of pillar stability, and so on.

This paper is intended to prepare the ground for a later exploitation of these attractive features. First, the general model is formulated. This involves discrete rock layers and permits the arbitrary specification of the thicknesses and properties of these laminae. Second, solutions resulting from some systematic variations in stratification are explored. Third, it is shown that in certain circumstances the model can be represented as a quasi-continuum. Last, some general solutions relating to the problem of seam extraction are introduced. Implicit in some of these solutions are attractive features that will promote the development later of the earlier-mentioned practical applications.

Frictionless Laminated Models

Formulation of the Basic Model

It is postulated that the rock mass is piece-wise homogeneous, consisting of homogeneous isotropic layers where the bedding planes, including the ground surface, are all parallel. The contact surfaces between beds are assumed to be free of shear stresses and of cohesion. The model is constructed to ensure continuity of stresses and normal displacements at the interfaces. As the layers cannot transmit tangential forces from one to another, it is sensible to model only horizontal stratification. A corollary to the assumptions of continuity in normal or vertical displacement and lack of cohesion is that the vertical *total* stress at every point of all interfaces should remain compressive throughout all phases of mining.

Sonntag, in a two-dimensional study published in 1957[13], examined the distribution of vertical stresses in a homogeneously laminated half-space due to a periodic normal pressure acting on its bounding surface. He obtained his results using the approximate theory of thin beams. The success of his approach prompted the formulation of the original frictionless model on the basis of the approximate theory of thin plates[1]. This feature of the early work is retained in the model described next.

Fig. 1 is a vertical section through the heterogeneous model showing that each lamina has distinct elastic properties and thickness. The Young's modulus, Poisson's ratio, and thickness of the j-th bed are E_j, ν_j, and t_j, respectively. The case where all layers have the same modulus and thickness will be referred to later as *homo-*

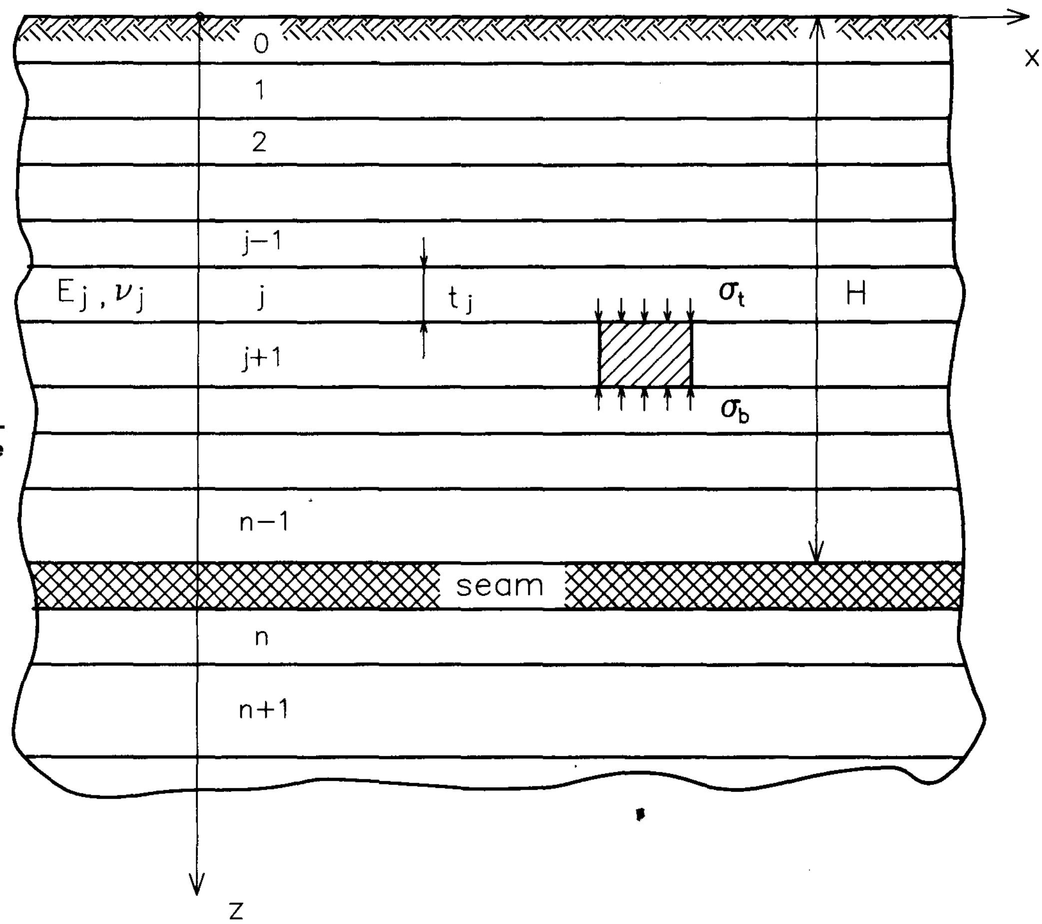

Fig. 1—Vertical section showing the stratification and the definition of notations

geneous stratification.

The thin-plate theory[14] defines the relationship between the deflection of the middle plane of a plate, w, and the transverse pressure acting in it, p, as

$$D\,\nabla^4 w(x,y) = p(x,y), \quad (1)$$

where D, in Love's terminology, is the 'flexural rigidity' of the plate:

$$D = E\,t^3/12\,(1-\nu^2) \quad (2)$$

and ∇^4 denotes throughout this paper the biharmonic operator in the horizontal plane:

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \quad (3)$$

In this definition the positive directions of pressure p and deflection w coincide. In obtaining (1) it was assumed that the bounding surfaces of the plate are free of shear stresses[14].

Throughout the paper, positive normal stresses and strains signify compression, and the positive z-axis points vertically downwards. These sign conventions necessitate that upward displacement is taken to be positive.

Turn now to the development of the frictionless laminated model. However, note first that the later discussion, unless it is indicated to the contrary, will refer to displacements and stresses *induced* by mining. It was observed earlier[15] that the induced system of forces is free of body forces. Thus, body forces can be disregarded in this paper. Also, almost everywhere a specific reference in notation to induced stresses can be omitted without fear of confusion.

The model is constructed by taking the appropriate steps to ensure that the vertical displacement and stress are continuous at the interfaces. According to Fig. 1, the transverse pressure acting on the $(j+1)$-th plate is

$$p_{j+1} = \sigma_b - \sigma_t. \quad (4)$$

Here σ_b and σ_t denote the induced vertical stresses at the bottom and the top of the $(j+1)$-th plate respectively. This definition ensures that positive p_{j+1} induces positive vertical displacement. To establish relationships between σ and w, note Love's observation[14] that the thickness t will not change in the first approximation provided no traction acts on the faces of the bent plate and the stretching of the middle plane is neglected. Thus, the change in thickness can be attributed solely to the effect of vertical stress, σ. More specifically, the reduction in the distance ½ $(t_j + t_{j+1})$ between the middle planes of laminae j + 1 and j is $w_{j+1} - w_j$. This change in distance is due to the strain σ_t/E_j in the lower half of the j-th bed and strain σ_t/E_{j+1} in the upper half of the $(j+1)$-th bed. Thus,

$$w_{j+1} - w_j = \frac{1}{2}\sigma_t\left(\frac{t_j}{E_j} + \frac{t_{j+1}}{E_{j+1}}\right), \quad (5)$$

from which

$$\sigma_t = 2\frac{E_j E_{j+1}}{k_j}(w_{j+1} - w_j). \quad (6)$$

An analogous argument yields

$$\sigma_b = 2\frac{E_{j+1}E_{j+2}}{k_{j+1}}(w_{j+2} - w_{j+1}). \quad \text{.....................} \quad (7)$$

In these expressions,

$$k_j = t_j E_{j+1} + t_{j+1} E_j. \quad \text{.................................} \quad (8)$$

The substitution of the results of (6) and (7) into (4), after some rearrangement, leads to

$$p_{j+1} = 2E_{j+1}\left[\frac{E_j}{k_j}(w_{j+2} - 2w_{j+1} + w_j) + \left(\frac{E_{j+2}}{k_{j+1}} - \frac{E_j}{k_j}\right)(w_{j+2} - w_{j+1})\right]. \quad \text{.............} \quad (9)$$

A close scrutiny of the reasoning leading to this result will show that the formulae in (5) to (9) ensure the continuity of the normal and shear stresses and of the vertical displacement across the connecting faces of rock beds. Note, however, that *no* continuity exists in the horizontal displacement at the same interfaces. This lack of continuity and the continuity of shear stresses are, of course, the consequences of the assumption that contact between neighbouring strata is free of shear stresses.

Now apply the relationship in (1) to the (j + 1)-th layer. To achieve this, substitute the expression for transverse pressure in (9) into the right-hand side of (1) and multiply both sides of the new equation by $t_{j+1}/2\,E_{j+1}$. The result is

$$\frac{t_{j+1}^4}{24(1 - \nu_{j+1}^2)}\nabla^4 w_{j+1} = \frac{1}{\delta_j + 1}\Delta^2 w_j + \left(\frac{\delta_{j+1}}{\delta_{j+1} + 1} - \frac{1}{\delta_j + 1}\right)\Delta w_{j+1}, \quad \text{................} \quad (10)$$

where

$$\delta_j = \frac{t_j E_{j+1}}{t_{j+1} E_j} \quad \text{...} \quad (11)$$

and Δ denotes the difference operator of the finite difference calculus; that is,

$$\Delta^r w_j = \Delta(\Delta^{r-1} w_j) = \Delta^{r-1} w_{j+1} - \Delta^{r-1} w_j$$
$$r = 1,2,\ldots \quad \text{..} \quad (12)$$

Finally, a simple re-arrangement of the terms gives

$$\delta_{j+1}w_{j+2} - (\delta_{j+1} + 1)\left[\frac{t_{j+1}^4}{24(1 - \nu_{j+1}^2)}\nabla^4 + \Delta\left(\frac{\delta_j}{\delta_j + 1}\right) + 1\right] w_{j+1} + \frac{\delta_{j+1} + 1}{\delta_j + 1} w_j = 0. \quad \text{........................} \quad (13)$$

This equation controls the heterogeneous frictionless laminated model featuring discrete layers. It is a partial differential equation with respect to the x,y variables, and a finite difference equation with regard to the variable z.

Systematic Stratifications

The controlling equation in (13) may simplify considerably if the stratification is assumed to have some regular features. Perhaps the most obvious of these is when

$$\delta_j = \delta = \text{constant}. \quad \text{..................................} \quad (14)$$

The substitution of this form into (13) leads to:

$$\delta w_{j+2} - (\delta + 1)\left[\frac{t_{j+1}^4}{24(1 - \nu_{j+1}^2)}\nabla^4 + 1\right] w_{j+1} + w_j = 0. \quad \text{..................................} \quad (15)$$

The relationship in (14) will be satisfied, for example, if

$$t_j = t_o\beta_1^j, \; E_j = E_o\beta_2^j, \; \beta_1 > 0, \; \beta_2 > 0, \quad \text{...........} \quad (16)$$

which give $\delta = \beta_2/\beta_1$. Here t_o, E_o, β_1, and β_2 are all given positive constants; where β_1 and β_2 are dimensionless, t_o and E_o have the dimensions of length and stress respectively.

An important further simplification is achieved if all laminae are taken to have the same thickness; that is, if $t_j = t_o = t$. This will be so when $\beta_1 = 1$. Clearly, it is permissible in this instance to omit the subscript 2 in (16) and simply use $E_j = E_o\beta^j$. Also, it seems convenient to ignore the variation in the value of Poisson's ratio; in other words, to postulate that $\nu_j = \nu$. (This is not a momentous simplification since Poisson's ratio appears to play only a secondary role in the approximate frictionless laminated model.) A rock mass having these features will be referred to later as the *uniformly stratified* rock mass. This idealized rock mass is mathematically significant because all the coefficients in its controlling equation are constants. This characteristic facilitates the analytical solution of (15).

Stratified Quasi-continuum

When the thickness of laminae in a *uniformly* stratified rock mass is small in relation to the typical linear dimensions of the practical problem under consideration, the finite differences in (10) can be approximated in terms of differential coefficients; that is,

$$\Delta w_{j+1} \simeq t\frac{\partial w}{\partial z}, \quad \Delta^2 w_j \simeq t^2\frac{\partial^2 w}{\partial z^2}. \quad \text{.....................} \quad (17)$$

Now, after employing these approximations, suppressing the subscripts, dividing both sides by t^2, and rearranging, the relationship in (10) becomes

$$\frac{1}{2}(\beta + 1)\lambda^2\nabla^4 w - \frac{\partial^2 w}{\partial z^2} - \frac{\beta - 1}{t}\frac{\partial w}{\partial z} = 0, \quad \text{........} \quad (18)$$

which is the partial differential equation purporting to describe the behaviour of a stratified mass. In this equation,

$$\lambda^2 = \frac{t^2}{12(1 - \nu^2)}. \quad \text{..} \quad (19)$$

This formulation sacrifices the specificity of individual laminae, although the resulting quasi-continuum retains the suppleness of the original model. The variation in Young's modulus is now described by a continuous function of z, which can be determined from the expression of vertical stress in (6) or (7). For example, from (6) the approximate relation between vertical stress and displacement is

$$\sigma_t = \frac{2\beta E_j}{\beta+1}\frac{w_{j+1} - w_j}{t} \simeq \frac{2\beta E_j}{\beta+1}\frac{\partial w}{\partial z}. \quad (20)$$

Take the exponential function

$$E_j = E_o\beta^{z/t}/\sqrt{\beta} \quad (21)$$

to describe the variation in E_j from layer to layer. This function gives, for the middle plane of the j-th bed (that is, for $z = (j+1/2)t$), a value that is identical to the Young's modulus obtained from (16) for the same lamina ($\beta_2 = \beta$). Accept that the vertical stress, σ, is proportional to the vertical strain $\epsilon = \partial w/\partial z$ and that the factor of proportionality is the effective Young's modulus, $E(z)$. It follows from (20) and (21) that

$$E(z) = \frac{2\sqrt{\beta}E_o}{\beta+1}\beta^{z/t} \quad (22)$$

and therefore,

$$\sigma(z) = E(z)\frac{\partial w}{\partial z}. \quad (23)$$

If the model is homogeneous (that is, if $\beta = 1$), the differential equation in (18) is identical to that derived in 1961 for the same type of stratification[1]. If, however, the value of the Young's modulus is not a constant, there appears to be a discrepancy in the coefficient of $\partial w/\partial z$. Also, in the earlier publication it was stated that E and t can be any differentiable function of z. The derivation here reveals, however, that the differential equation in (18) will yield an accurate approximation to the result derived from (13) only if the stratification is uniform ($t_j = t$). In the case of uniform stratification and if β does not deviate much from unity, the basic equation of the early work[1] and that in (18) become identical.

Status of the Models

It can be argued that models incorporating the assumption of frictionless contact between neighbouring strata are far-fetched and unrealistic, and that they do not have a place in modern rock-mechanics literature. This is a persuasive argument since there are numerical techniques that can handle stratified rocks without employing the simplifications arising from shear-free contacts[11,12].

The present intention to revive these models is *not* motivated by a plan to apply them in situations where the use of more complex models is justified. Instead, the aim is to seek a tool that facilitates the approximate solution, in an easy-to-apply manner, of practical problems associated with the extraction of coal seams.

It is not too difficult to accept the notion that such a tool would take the form of a simple mathematical model that contains a few physical parameters relating to the rock mass and obeys most of the fundamental relations of mechanics, such as a reasonable set of stress–strain relations, equations of equilibrium, and so on. Whether or not such a model has practical utility should be decided upon in the light of its demonstrated ability, or the lack of it, to describe real phenomena with acceptable accuracy.

Most of the potential weaknesses of the models discussed in this paper can be attributed to two sources. First, the models may fail because of the assumptions inherent in the idea that the behaviour of a pile of perfectly smooth elastic plates closely resembles that of a stratified rock mass. Second, the models may prove inadequate because the mathematical abstraction used here relies on the theory of thin plates.

While no rock mass is perfectly elastic, the concept represents the simplest approach to the modelling of solids. Linear models remain useful in the solution of many practical problems, as evidenced by the relative success of subsidence predictions based on influence functions[2–4,16].

Perfectly smooth surfaces do not exist in nature, and the idea of frictionless interfaces between plates seems untenable. But experimental evidence suggests that some sliding between rock beds does occur in mines. In view of these conflicting facts, it can be concluded that the rock mass is neither a continuum, nor does it permit sliding at all contact surfaces between strata. Moreover, where sliding does occur, it is not free of restrictions caused by friction. These observations suggest that a frictionless laminated model may, at best, describe the behaviour of a stratified mass in some coarse or approximate sense, but its laminae are unlikely to coincide on a one-to-one basis with the strata in the field.

The use of the theory of thin plates is not essential to the construction of the model, as was shown by Wardle[11]. However, this theory leads to considerable simplifications and produces features that much enhance the utility of the model. In view of these and the earlier arguments concerning the fundamentally approximate nature of *any* frictionless laminated model, it seems reasonable to retain, at least for the present, the thin-plate theory. Sceptics should remember that the history of science, especially that of engineering science, is full of simplified models that have been used, and are still being used, successfully.

Mining of a Single Seam: Basic Solutions

Ever since the pioneering work of Hackett[5,6], displacement discontinuities have been employed with success in the solution of problems involving tabular excavations. The displacement discontinuity in this context involves a jump in displacement when moving from the roof to the floor of a tabular excavation. Since the application of frictionless laminated models is restricted to horizontal stratification, the only significant component of displacement discontinuity in this discussion is the convergence of the roof and floor; that is,

$$s = w^+ - w^-, \quad (24)$$

where w^+ and w^- are the vertical displacements of the immediate floor and roof planes respectively.

The time-honoured method of solving problems related to horizontal tabular excavations expresses the displacements and stresses in terms of the convergence in the seam[1,2,15]. An example of such a relationship is

$$w(x,y,z) = {}_A\!\int s(\xi,\eta)\, W[(x-\xi), (y-\eta), z]\mathrm{d}A. \quad (25)$$

In this integral A denotes the area in the seam where convergence occurs. The influence function W gives the vertical displacement at point x,y,z, induced by a unit volume of convergence centred at point ξ,η in the seam. Other influence functions exist for other components of displacement, strain, and stress.

Usually the solution to a problem involves, first, the derivation of the influence functions and then the determination of the convergence distribution. It should be noted that in the present case W is axially symmetric with respect to the vertical axis situated at ξ,η. In the following sections, solutions are presented initially for models consisting of discrete layers and then for models based on quasi-continua.

Uniformly Stratified Rock Mass

It will be convenient first to deal with the less general model involving uniform stratification and then to discuss the general case pertaining to arbitrary stratification. In the interest of conciseness, however, some of the boundary conditions of the problem are formulated in general terms so as to avoid later repetition of the arguments.

Since the faces of all laminae in the frictionless laminated model are free of shear, it is necessary to satisfy only four boundary conditions. The first two of these ensure, in turn, that the ground surface is free of normal stresses and that the normal stress is continuous across the plane of the seam. These two will be dealt with in general form. The third condition guarantees that, at infinite depth, the vertical displacement becomes zero. The fourth condition is specific to the problem under consideration.

According to Fig. 1, for the stratum that is exposed at the ground surface j = 0. Clearly, the upper face of this bed should be free of vertical stress. Thus, the lower half of lamina j = 0 is loaded by transverse pressure p_o, which is due only to the interaction with the layer beneath it (j = 1). Hence, from (4) and (7)

$$p_o = \sigma_b = \frac{2E_1 E_o}{k_o}(w_1 - w_o), \quad \text{......} \quad (26)$$

which, when substituted into (1), yields

$$\frac{t_o^3}{24(1-\nu_o^2)}\nabla^4 w_o = \frac{E_1}{k_o}(w_1 - w_o). \quad \text{......} \quad (27)$$

Care needs to be taken when dealing with the relationships in (4) to (11) because they were formulated with respect to layer j + 1.

As shown in Fig. 1, the roof bed is numbered j = n − 1, and the floor bed is j = n. Let the transverse pressure acting on the roof bed be $p_{n-1} = \sigma - \sigma_t$ and that on the floor stratum be $p_n = \sigma_b - \sigma$, where σ is some unknown vertical stress. These definitions of transverse pressure ensure that the normal stress remains continuous when moving from the roof to the floor. Components σ_t and σ_b can be obtained from (6) and (7):

$$\sigma_t = \frac{2E_{n-1}E_{n-2}}{k_{n-2}}(w_{n-1} - w_{n-2}),$$

$$\sigma_b = \frac{2E_{n+1}E_n}{k_n}(w_{n+i} - w_n). \quad \text{......} \quad (28)$$

Now define the controlling equations of the roof and floor plates by substituting p_{n-1} and p_n into (1):

$$\frac{t_{n-1}^3 E_{n-1}}{12(1-\nu_{n-1}^2)}\nabla^4 w_{n-1} = \sigma - \frac{2E_{n-1}E_{n-2}}{k_{n-2}} \times (w_{n-1} - w_{n-2}) \quad \text{......} \quad (29)$$

$$\frac{t_n^3 E_n}{12(1-\nu_n^2)}\nabla^4 w_n = \frac{2E_{n+1}E_n}{k_n}(w_{n+1} - w_n) - \sigma. \quad \text{...} \quad (30)$$

To eliminate σ, add these equations to obtain

$$\frac{1}{24}\nabla^4\left(\frac{t_n^3 E_n}{1-\nu_n^2}w_n + \frac{t_{n-1}^3 E_{n-1}}{1-\nu_{n-1}^2}w_{n-1}\right) = \frac{E_{n+1}E_n}{k_n}(w_{n+1} - w_n) - \frac{E_{n-1}E_{n-2}}{k_{n-2}} \times (w_{n-1} - w_{n-2}). \quad \text{......} \quad (31)$$

Next relate the convergence of the immediate roof and floor, which is defined in (24), to the displacement of the middle planes of these strata. Clearly, function s is given by the convergence of the middle planes of the floor and roof beds reduced by the change in distance caused by stress σ. Thus,

$$s = w_n - w_{n-1} - \frac{k_{n-1}\sigma}{2E_n E_{n-1}}, \quad \text{......} \quad (32)$$

where σ can be obtained from (29) or (30).

Having laboured to obtain the results in (26) to (32), return now to the uniformly stratified model; that is, to the case where

$$t_j = t_o = t, \quad E_j = E_o\beta^j, \quad \nu_j = \nu, \quad \delta = \beta. \quad \text{......} \quad (33)$$

The substitution of these results into (15) yields the general equation of this model:

$$\beta w_{j+2} - (\beta+1)\left[\frac{t^4}{24(1-\nu^2)}\nabla^4 + 1\right] \times w_{j+1} + w_j = 0. \quad \text{......} \quad (34)$$

To facilitate the solution of this equation it will be convenient to employ integral transforms. It was noted that the influence function W is axially symmetric. This observation suggests that in some circumstances the Hankel transform of zero order can be used with advantage. In this case, the transform of w is given by[17]

$$\bar{w}(\Psi,z) = \int_0^\infty w(r,z)\, r J_o(\Psi r)\, dr, \quad \text{......} \quad (35)$$

where $J_o(.)$ is the zero-order Bessel function of the first kind. A reasonably general solution can be obtained through the use of the double Fourier transform[1,2,11]:

$$\bar{w}(\Psi_1,\Psi_2,z) = \int_{-\infty}^\infty\int_{-\infty}^\infty w(x,y,z)\, e^{i(\Psi_1 x + \Psi_2 y)}\, dx\, dy. (36)$$

Also, useful two-dimensional solutions are yielded by the Fourier cosine transform, which is defined by[17]

$$\bar{w}(\Psi,z) = \int_0^\infty w(x,z)\cos x\Psi\, dx. \quad \text{......} \quad (37)$$

Multiply (34) by $rJ_0(\Psi r)$ and integrate with respect to r from zero to infinity. The result is

$$\beta \bar{w}_{j+2} - (\beta+1)\left[\frac{\Psi^4 t^4}{24(1-\nu^2)} + 1\right]\bar{w}_{j+1} + \bar{w}_j = 0, \quad (38)$$

which was obtained after noting that w_j has some simple symmetry properties and its value, together with its first three derivatives with respect to r, approach zero as the radius is increased to infinity.

It is interesting to note that the double Fourier transform of (34) also yields (38) provided w_j and its first three partial derivatives with respect to x and y are zero at positive and negative infinities. In this case, the notation

$$\Psi^2 = \Psi_1^2 + \Psi_2^2 \quad (39)$$

is implied. Similarly, the application of the cosine transform to (34) leads also to (38), provided w_j is symmetric with respect to x and the function itself, together with its first three derivatives with respect to the same variable, are zero at $x = \infty$. This observation means that the solution of (38) is the transform of the vertical displacement, *irrespective* of whether the Hankel, the double Fourier, or the cosine transform is employed. Of course, to obtain the displacement function itself, it is necessary to use the appropriate inversion formula.

To solve the finite difference equation in (38), postulate that $\bar{w}_j = C(\Psi)\,\alpha^j$, which, when substituted into (38), yields

$$[\beta\alpha^2 - (\beta+1)a\alpha + 1]\,C(\Psi)\,\alpha^j = 0, \quad (40)$$

where

$$a = \frac{\Psi^4 t^4}{24(1-\nu^2)} + 1 \geq 1. \quad (41)$$

This equation will be satisfied only if

$$\beta\alpha^2 - (\beta+1)a\alpha + 1 = 0. \quad (42)$$

There are advantages in replacing α by another variable κ. Let

$$\alpha = \kappa/\beta, \quad (43)$$

which, when substituted into (42), indicates that κ is defined by the quadratic equation

$$\kappa^2 - (\beta+1)a\kappa + \beta = 0, \quad (44)$$

the roots of which are

$$\kappa_1 = \frac{1}{2}\left[(\beta+1)a + \sqrt{(\beta+1)^2a^2 - 4\beta}\right],$$

$$\kappa_2 = \frac{1}{2}\left[(\beta+1)a - \sqrt{(\beta+1)^2a^2 - 4\beta}\right]. \quad (45)$$

These roots are both real and positive as long as $\beta > 0$. They satisfy the following relationships:

$$\kappa_1 + \kappa_2 = (\beta+1)a > 1, \quad \kappa_1 - \kappa_2 \geq 0,$$
$$\kappa_1\kappa_2 = \beta > 0 \quad (46)$$

and

$$(\kappa_k \pm \beta)(\kappa_k \pm 1) = (a \pm 1)(\beta + 1)\kappa_k \geq 0$$
$$k = 1, 2, \quad (47)$$

where either all the upper or all the lower signs are to be used. Since $a \geq 1$, the right-hand side of (47) is non-negative, which requires that *either* $\kappa_k - \beta \geq 0$ and $\kappa_k - 1 \geq 0$ *or* $\kappa_k - \beta \leq 0$ and $\kappa_k - 1 \leq 0$. Brief reflection reveals that both roots cannot be either larger or smaller than unity at the same time, but the following inequalities must apply:

$$\text{(i) } 0 < \beta \leq 1 \quad \kappa_1 \geq 1, \quad \kappa_2 \leq \beta,$$
$$\alpha_1 \geq 1/\beta, \quad \alpha_2 \leq 1, \quad (48)$$

$$\text{(ii) } \beta \geq 1 \quad \kappa_1 \geq \beta, \quad \kappa_2 \leq 1,$$
$$\alpha_1 \geq 1 \quad \alpha_2 \leq 1/\beta. \quad (49)$$

In the light of these limits, the solution of (38) for the roof strata takes the form

$$\bar{w}_j = C_1(\Psi)\,\alpha_1^j + C_2(\Psi)\,\alpha_2^j \quad j = 0, 1, \ldots (n-1) \quad (50)$$

and that for the strata in the floor

$$\bar{w}_j = C_0(\Psi)\,\alpha_2^j \quad j = n, (n+1), (n+2), \ldots \quad (51)$$

The latter of these solutions reduces to zero as the value of j increases to infinity. Thus, this expression automatically satisfies the remote boundary condition mentioned earlier.

Two of the three unknown C functions can be determined from the boundary conditions as defined by (27) and (31). Substitute from (33) the parameters corresponding to uniform stratification into (27) and (31), and take the transform of both sides of these equations. The results of these manipulations are as follows:

$$[(\beta + 1)(a - 1) + \beta]\,\bar{w}_0 - \beta\,\bar{w}_1 = 0, \quad (52)$$

$$\beta^2\bar{w}_{n+1} - \beta[(\beta+1)(a-1)+\beta]\bar{w}_n -$$
$$[(\beta+1)(a-1)+1]\bar{w}_{n-1} + \bar{w}_{n-2} = 0. \quad (53)$$

After the substitution of the appropriate transforms from either (50) or (51), these equations yield

$$C_0(\Psi) = -\left[\left(\frac{\kappa_1}{\kappa_2}\right)^n - 1\right] C_2(\Psi),$$

$$C_1(\Psi) = \frac{\kappa_1 - 1}{1 - \kappa_2}\,C_2(\Psi) \quad (54)$$

and, in turn, these provide to the displacement transforms

$$\bar{w}_j = \left[\frac{\kappa_1 - 1}{1 - \kappa_2}\alpha_1^j + \alpha_2^j\right] C_2(\Psi)$$
$$j = 0, 1, \ldots, (n-1) \quad (55)$$

$$\bar{w}_j = -\left[\left(\frac{\kappa_1}{\kappa_2}\right)^n - 1\right]\alpha_2^j\,C_2(\Psi)$$
$$j = n, (n+1), (n+2), \ldots \quad (56)$$

The next step is to determine the transform of the convergence distribution using the result in (32). First note from (8) and (33) that, for uniform stratification,

$$k_{n-1} = tE_0(\beta+1)\beta^{n-1}. \quad (57)$$

Now substitute this formula and the expression for σ obtained from (29) into (32). Some algebraic manipulations then yield

$$s(x,y) = \frac{1}{\beta}\left\{\beta w_n - (\beta+1)\left[\frac{t^4}{24(1-\nu^2)}\nabla^4 + 1\right]\times w_{n-1} + w_{n-2}\right\}. \quad (58)$$

A quick comparison with the result in (34) reveals that the right-hand side of this expression would be zero but for the displacement discontinuity between rock beds (n – 1) and n. Thus, the definition of convergence in (58) confirms that the equation in (34) is valid for all those beds where the vertical displacement is continuous at both the top and the bottom interfaces.

Now take the transform of both sides of (58). This step yields

$$\bar{s} = \frac{1}{\beta}[\beta \bar{w}_n - (\beta+1)a\bar{w}_{n-1} + \bar{w}_{n-2}], \quad (59)$$

regardless of whether the Hankel, the double Fourier, or the Fourier cosine transform is employed. Here, a is defined in (41). Substitute for the displacement transforms from (55) and (56), and then rearrange to obtain

$$\bar{s} = -\frac{\kappa_1 - \kappa_2}{1 - \kappa_2}\alpha_1^n C_2(\Psi). \quad (60)$$

The only remaining unknown function is $C_2(\Psi)$, which is to be determined from the remaining boundary condition at the seam horizon. The immediate goal is to derive the axially symmetric influence function W, which is the displacement caused by a unit volume of convergence at a 'point' in the seam.

First, for reasons that will become obvious later, the displacement field due to uniform convergence s_m within radius R is derived. Thus, postulate that

$$\begin{aligned} 0 \le r \le R \qquad & s(r) = s_m, \\ r \ge R \qquad & s(r) = 0. \end{aligned} \quad (61)$$

Therefore, the Hankel transform of the convergence distribution is given by

$$\bar{s}(\Psi) = s_m \int_0^R r J_0(r\Psi)\,dr = \frac{s_m R}{\Psi} J_1(R\Psi). \quad (62)$$

This result, when substituted into (60), yields

$$C_2(\Psi) = -\frac{s_m R}{\Psi}\frac{1-\kappa_2}{\kappa_1 - \kappa_2}\kappa_2^n J_1(R\Psi), \quad (63)$$

which, in turn, defines with the aid of (55) and (56) the displacement field. After employing the inversion formula, the displacement in the roof strata is

$$w_j(r,R) = -s_m R \int_0^\infty \frac{\kappa_2^n}{\kappa_1 - \kappa_2}[(\kappa_1 - 1)\alpha_1^j + (1-\kappa_2)\alpha_2^j]\times J_1(R\Psi)J_0(r\Psi)\,d\Psi \quad j = 0, 1, \ldots (n-1) \quad (64)$$

and in the floor it becomes

$$w_j(r,R) = s_m R \int_0^\infty \frac{1-\kappa_2}{\kappa_1 - \kappa_2}(\kappa_1^n - \kappa_2^n)\alpha_2^j J_1(R\Psi)J_0(r\Psi)\,d\Psi \quad j = n, (n+1), \quad (65)$$

To derive the influence function due to a point convergence, assume that R is infinitesimal, and it therefore suffices to retain the first-order term in the expansion of the first-order Bessel function; that is, $J_1(R\Psi) = \frac{1}{2}R\Psi$. Also, let $\pi R^2 = \Delta A$. Now substitute these results into (64) and (65), and note the definition of the influence function in (25).

These preparations produce the following influence functions:

$$W_j(r,jt) = -\frac{1}{2\pi}\int_0^\infty \frac{\kappa_2^n}{\kappa_1 - \kappa_2}[(\kappa_1 - 1)\alpha_1^j + (1-\kappa_2)\alpha_2^j]\times \Psi J_0(r\Psi)\,d\Psi, \quad j = 0, 1, 2, \ldots (n-1) \quad (66)$$

$$W_j(r,jt) = \frac{1}{2\pi}\int_0^\infty \frac{1-\kappa_2}{\kappa_1 - \kappa_2}(\kappa_1^n - \kappa_2^n)\Psi\alpha_2^j J_0(r\Psi)\,d\Psi, \quad j = n, (n+1), \quad (67)$$

These formulae permit the computation of the vertical displacement w at point x,y,jt:

$$w(x,y,jt) = \iint_A s(\xi,\eta)\,W_j(r,jt)\,d\xi d\eta, \quad (68)$$

where $s(\xi,\eta)$ is a given distribution of convergence within area A and $r = [(x-\xi)^2 + (y-\eta)^2)]^{1/2}$.

Some Properties of Influence Functions

The results presented here might be expected to conform to a basic *hypothesis* that was formulated many years ago[2]. It was postulated that, if a horizontal seam of uniform thickness is extracted over a large and increasing area, the vertical displacement in the centre of the area should approach, as mining spreads, zero below the seam and the value of roof and floor convergence above the seam. Thus, this hypothesis requires that, at the centre of the mined-out area (that is, at $r = 0$), (64) and (65) should yield $w_j(0,R) \to -s_m$ and $w_j(0,R) \to 0$ respectively as R approaches infinity.

The vertical displacement in *any* linearly elastic model due to a uniform convergence, s_m, within radius R—see (61)—can be expressed in the following form:

$$w(r,R) = s_m R \int_0^\infty f_k(\Psi)J_1(R\Psi)J_0(r\Psi)\,d\Psi, \quad k = 1,2. \quad (69)$$

which reduces to

$$w(0,R) = s_m R \int_0^\infty f_k(\Psi)J_1(R\Psi)\,d\Psi \quad (70)$$

at the centre of the mined-out area. In these expressions, $f_1(\Psi)$ and $f_2(\Psi)$ refer to the roof and floor strata respectively. The result in (69) is a generalization of those given in (64) and (65). Next, determine the value of $w(0,R)$ in (70) as R approaches infinity.

Tranter[17], using some results he attributes to Willis, derived an asymptotic formula for the type of integral exemplified by (70). He states that, for large values of R,

$$\int_0^\infty f(\Psi)J_1(R\Psi)\,d\Psi = \frac{f(0)}{R} + \frac{f'(0)}{R^2} - \frac{1}{2}\frac{f'''(0)}{R^4} + \ldots \quad (71)$$

This asymptotic expansion leads to the following limit:

$$\lim_{R \to \infty} \left[R \int_0^\infty f(\Psi)\, J_1(R\Psi)\, d\Psi\right] = f(0). \quad \text{..........} \quad (72)$$

The basic hypothesis requires that the relationships $w_j(0,\infty) = -s_m f_1(0) = -s_m$ and $w_j(0,\infty) = s_m f_2(0) = 0$ should hold above and below the seam respectively. Clearly, on the basis of (70) and (72), this will be so if, and only if,

$$f_1(0) = 1 \text{ and } f_2(0) = 0. \quad \text{..........................} \quad (73)$$

Now examine the uniformly stratified model in the light of these conclusions.

According to (64) and (65), function $f(\Psi)$ takes different forms in the roof and floor. The function in the roof is

$$f_1(\Psi) = \frac{\kappa_2^n}{\kappa_1 - \kappa_2}\left[(\kappa_1 - 1)\alpha_1^j + (1 - \kappa_2)\alpha_2^j\right]$$

$$j = 0, 1, \ldots (n-1) \quad \text{............................} \quad (74)$$

and in the floor it is defined by

$$f_2(\Psi) = \frac{1 - \kappa_2}{\kappa_1 - \kappa_2}(\kappa_1^n - \kappa_2^n)\,\alpha_2^j \qquad j = n, (n+1), \ldots \quad (75)$$

Next, analyse the behaviour of these functions. The inequalities in (48) and (49) warn that it will be necessary to distinguish the case of $0 < \beta \leq 1$ from that when $\beta \geq 1$. Thus, for $\Psi = 0$, (45) yields the following values:

(i) $0 < \beta \leq 1$ $\quad \kappa_1 = 1,\ \kappa_2 = \beta,\ \alpha_1 = 1/\beta,\ \alpha_2 = 1$

$$f_1(0) = \beta^n, \quad f_2(0) = 1 - \beta^n, \quad \text{......} \quad (76)$$

(ii) $\beta \geq 1$ $\quad \kappa_1 = \beta,\ \kappa_2 = 1,\ \alpha_1 = 1,\ \alpha_2 = 1/\beta$

$$f_1(0) = 1, \quad f_2(0) = 0. \quad \text{..............} \quad (77)$$

These results reveal that the following relationship holds regardless of the value of β:

$$w_{n+i}(0,\infty) - w_j(0,\infty) = s_m \qquad \begin{array}{l} i = 0, 1, 2, \ldots \\ j = 0, 1, 2, \ldots \\ (n-1). \quad \text{...............} \quad (78) \end{array}$$

This means that the *relative* vertical displacement between a pair of points, provided one of them is located in the floor and the other in the roof, equals the convergence. At the same time, the requirements in (73) are satisfied *only* if $\beta \geq 1$. Consequently, in the case of $0 < \beta < 1$, the basic hypothesis is violated and there might be justification for disregarding in the sequel cases corresponding to such values of β. However, in contrast to this view, it may be argued that values of β less than but close to unity could be used to model, at least as a first approximation, rock behaviour exhibiting some permanent dilatancy.

Heterogeneous Stratification

It is assumed here that no rule controlling the parameters of stratification (i.e. bed thickness, Young's modulus, etc.) is known to apply to some part of the rock mass. Therefore, to obtain a solution, it is necessary to prescribe values of these geological variables individually for each stratum. Since this cannot be done for an infinite number of layers, clearly only a finite portion of the mass can be treated in such a general manner.

Assume that the parameters of stratification are specified individually for m contiguous layers forming a stack that lies beneath the uppermost p beds. Thus, physical parameters are prescribed independently for layers j = p, (p + 1), ... (p + m − 1), where p and m are positive integers. While the value of p may be zero, obviously that of m is necessarily non-zero. In practice, often the most easily justified assumption is, in fact, to take p = 0.

Two fundamental relationships play important roles in this general case. The first of these defines the convergence or divergence between layers (j − 1) and j, and can be obtained from (32) and (29) after putting n = j:

$$s(x,y) = \frac{1}{\delta_{j-1}}\Bigg\{\delta_{j-1} w_j - (\delta_{j-1} + 1)\left[\frac{t_{j-1}^4}{24(1 - \nu_{j-1}^2)}\nabla^4 + \Delta\left(\frac{\delta_{j-2}}{\delta_{j-2} + 1}\right) + 1\right] w_{j-1} + \frac{\delta_{j-1} + 1}{\delta_{j-2} + 1} w_{j-2}\Bigg\}. \quad \text{.............................} \quad (79)$$

This formula simplifies to that in (58) if the stratification is uniform ($\delta_i = \beta$) and j = n. Also, it confirms that the basic equation in (13) ensures the continuity of vertical displacement. This is so because obviously $s(x,y) \equiv 0$ if w_{j-2}, w_{j-1}, and w_j satisfy (13).

The second basic relationship ensures the continuity of the induced vertical stress across the interface between laminae (j − 1) and j. This result is obtained by reformulating (31):

$$\frac{\delta_{j-1}\delta_j}{\delta_j + 1} w_{j+1} - \left[\frac{\delta_{j-1} t_j^4}{24(1 - \nu_j^2)}\nabla^4 + \frac{\delta_{j-1}\delta_j}{\delta_j + 1}\right] w_j - \left[\frac{t_{j-1}^4}{24(1 - \nu_{j-1}^2)}\nabla^4 + \frac{1}{\delta_{j-2} + 1}\right] w_{j-1} + \frac{1}{\delta_{j-2} + 1} w_{j-2} = 0. \quad \text{...............................} \quad (80)$$

Postulate that the stack of m laminae is in direct contact at its top and at its bottom with uniformly stratified rock masses of the type for which solutions were derived earlier in this paper. The stack may or may not enclose the coal seam that is to be mined. Assume first that no mining takes place within the stack. In this instance $s(x,y) \equiv 0$ for all interfaces internal to the stack, and (13) must be valid for j = (p + 1), (p + 2), ... (p + m − 2) giving (m − 2) equations. Furthermore, to ensure the continuity of vertical stress at the top and bottom of the stack, (80) is employed for j = p and j = (p + m). Two more equations result from the condition of continuity of displacement at the interfaces between the stack and the strata below and above it. Naturally, as was done earlier, the conditions at the ground surface, at infinite depth, and across the seam must also be satisfied.

If the stack does contain the seam to be mined, the scheme of solution is altered slightly. Now the stack contains laminae j = p, (p + 1), ... (n − 1), n, (n + 1), ... (p + m − 1). In this case (13) controls the behaviour of only j = (p + 1), ... (n − 2), (n + 1), ... (p + m

– 2), giving (m – 4) equations. An additional three equations follow from (80) and are obtained by putting j = p, n, (p + m). Two more equations arise from the continuity of displacements between the stack and the neighbouring strata above and below it. A single equation results from the existence of convergence between beds (n – 1) and n, and can be obtained from (79) by putting j = n. Of course, over and above these equations, the usual conditions at the surface and at great depth must also be complied with.

Almost all the equations in both of these systems contain the biharmonic operator ∇^4, and all of them must be satisfied simultaneously at all points of the xy plane. To handle such a system of equations it is necessary to turn to integral transforms to eliminate ∇^4 and the horizontal coordinates. The transformed forms of the equations are the same regardless whether Hankel, double Fourier, or Fourier cosine transforms are employed. The unknowns in the system of equations, for a given value of the transformation variable Ψ, are the transforms of the displacement components; that is, $\bar{w}_j$ for j = p, (p + 1), ... (p + m – 1) and the unknown functions that are introduced as a result of the contact with the uniformly stratified regions of the rock mass above and below the stack of m layers—see (50) and (51).

The elements of the coefficient matrix of the system and the unknowns themselves are continuous functions of the transformation variable Ψ. Dependence on Ψ (or on Ψ_1 and Ψ_2 if the double Fourier transform in (36) is used) may arise also through the transform of $s(x,y)$. To obtain ultimately the required displacement or stress components, it is necessary to solve repeatedly the system of equations for as many values of Ψ (or for Ψ_1 and Ψ_2) that are necessary to facilitate the application of the appropriate inversion formula to those transforms $\bar{w}_j$ that are of particular interest[12]. No attempt is made here to discuss the numerical scheme that would ensure that the integrations in the inversions yield results of acceptable accuracy.

The desire to keep this section concise prevented a more extensive discussion of the solution of cases involving heterogeneous stratification. Most practical cases can be solved on the basis of the principles elaborated here and in the earlier parts of the paper. For example, the instance when the stack is exposed on the surface can be solved by recognizing that, in this instance, p = 0 and the Young's moduli with negative subscript are to be equated to zero. Situations involving continuous media outside the stack can also be handled with relative ease. This is achieved by using the transforms of the normal displacement and stress derived from the continuous models at the interfaces between the stack and the surrounding rocks.

Uniformly Stratified Quasi-continuum

In certain circumstances models containing distinct layers can be replaced by some equivalent, ostensibly continuous media. The equation in (18) controls the behaviour of such a quasi-continuum, which professes to be equivalent to a uniformly stratified mass.

To obtain a reasonably general solution, take the transform of (18):

$$\frac{1}{2}(\beta+1)\lambda^2\Psi^4\bar{w} - \frac{d^2\bar{w}}{dz^2} - \frac{\beta-1}{t}\frac{d\bar{w}}{dz} = 0, \quad \text{........} \quad (81)$$

where t is the effective lamination thickness and constants $\beta = \beta_2$ and λ are as defined earlier in (16) and (19). The general solution of this equation, designed to handle both the roof and floor strata, is

$$\bar{w} = C_{2k-1}(\Psi)e^{\gamma_1 z/t} + C_{2k}(\Psi)e^{-\gamma_2 z/t}, \quad k = 1,2, \text{ .. } (82)$$

where k = 1 corresponds to the overburden and k = 2 represents the rock mass in the floor. This expression contains the parameters

$$\gamma_1 = \frac{1}{2}[\sqrt{(\beta-1)^2 + 2(\beta+1)t^2\lambda^2\Psi^4} - (\beta-1)] \geq 0,$$

$$\gamma_2 = \frac{1}{2}[\sqrt{(\beta-1)^2 + 2(\beta+1)t^2\lambda^2\Psi^4} + (\beta-1)] \geq 0. \quad (83)$$

The vertical stress is given by the definition in (23), which when applied to (82) produces

$$\bar{\sigma} = E(z)\,[\gamma_1 C_{2k-1}(\Psi)e^{\gamma_1 z/t} - \gamma_2 C_{2k}(\Psi)e^{-\gamma_2 z/t}]/t. \text{ .. } (84)$$

The Young's modulus, $E(z)$ in this expression, is defined by (22).

The boundary conditions, which ensure a stress-free surface at $z = 0$, zero displacement and stress at $z = \infty$, continuity of vertical stress at $z = H$, and a given convergence distribution at the seam horizon ($z = H$), lead to the following results:

$$C_2(\Psi) = -\gamma_1 \exp(-\gamma_1 n)\bar{s}(\Psi)/(\gamma_1 + \gamma_2),$$
$$n = H/t, \quad \text{..} \quad (85)$$

and

$$C_1(\Psi) = \frac{\gamma_2}{\gamma_1}C_2(\Psi), \; C_3(\Psi) = 0,$$
$$C_4(\Psi) = -[e^{(\gamma_1+\gamma_2)n} - 1]\,C_2(\Psi). \quad \text{................} \quad (86)$$

The elementary vertical displacement and stress corresponding to a face element are obtained by postulating a constant convergence, s_m, within an infinitesimal radius, R. The transform of the convergence is specified by (61) and (62) which, as R is small, reduces to

$$\bar{s}(\Psi) = \frac{1}{2}s_m R^2 = \frac{s_m \Delta A}{2\pi}. \quad \text{............................} \quad (87)$$

This result, when substituted into (85) and (86), and then into (82), leads ultimately to the following influence functions:

$$W(r,z) = \frac{(-1)^k}{2\pi}\int_0^\infty \Psi f_k(\Psi,z)\,J_0(r\Psi)\,d\Psi, \quad \text{........} \quad (88)$$

where the value of k was defined in conjunction with (82). Functions $f_1(\Psi,z)$ and $f_2(\Psi,z)$ take the following forms:

$$f_1(\Psi,z) = \frac{1}{\gamma_1+\gamma_2}(\gamma_2 e^{\gamma_1 z/t} + \gamma_1 e^{-\gamma_2 z/t})\,e^{-\gamma_1 n}, \quad \text{......} \quad (89)$$

$$f_2(\Psi,z) = \frac{\gamma_1}{\gamma_1+\gamma_2}(e^{\gamma_2 n} - e^{-\gamma_1 n})\,e^{-\gamma_2 z/t}. \quad \text{............} \quad (90)$$

Next, postulate that the convergence is constant and equal to s_m over a circular area of radius R and zero outside that region. The vertical displacement, $w(R,r,z)$, at $x = y = 0$ can be obtained through the integration of the influence functions

$$w(R,0,z) = s_m \int_0^{2\pi}\int_0^{R} rW(r,z)\mathrm{d}r\,\mathrm{d}\omega = (-1)^k s_m R \int_0^{\infty} f_k(\Psi,z)J_1(R\Psi)\,\mathrm{d}\Psi. \quad \text{.................} \quad (91)$$

Now, let R approach infinity and derive the corresponding limit, employing the result in (72):

$$w(\infty,0,z)/s_m = (-1)^k \lim_{R\to\infty} R \int_0^{\infty} f_k(\Psi,z)J_1(R\Psi)\mathrm{d}\Psi = (-1)^k f_k(0,z). \quad \text{........................} \quad (92)$$

The appropriate values of $f_k(0,z)$ are as follows:

$$\begin{array}{lll} 0 < \beta \le 1 & f_1(0,z) = e^{-(1-\beta)n} & f_2(0,z) = 1 - e^{-(1-\beta)n}, \\ \beta \ge 1 & f_1(0,z) = 1, & f_2(0,z) = 0. \quad \text{.........} \quad (93) \end{array}$$

These results reveal the same situation as that indicated by the data in (76) and (77). The vertical displacement in (92) equals s_m above the seam and zero below it if, and only if, $\beta \ge 1$. Also, the relative displacement of a pair of points, one of which is below and the other is above the seam, equals s_m regardless of the value β.

Influence Functions for Surface Subsidence

The simplest and the most obvious application of the results in this paper involves the prediction of surface subsidence. In this section, the subsidence influence functions for the uniformly stratified model are analysed with a view to comparing the distinct plate and quasi-continuum versions.

First, examine the distinct plate solution on the basis of the result in (66). Substitute j = 0 to obtain

$$W_0(r,0) = -\frac{1}{2\pi}\int_0^{\infty} \Psi\, \kappa_2^n J_0(r\Psi)\,\mathrm{d}\Psi \quad \text{................} \quad (94)$$

which, after multiplication by $s_m \Delta A$, provides the elementary subsidence distribution induced by a seam element.

$$\Delta w_0(r) = -\frac{s_m \Delta A}{2\pi}\int_0^{\infty} \Psi\, \kappa_2^n J_0(r\Psi)\,\mathrm{d}\Psi. \quad \text{............} \quad (95)$$

It will be more convenient to use dimensionless coordinates. Introduce

$$\rho = r/H,\ \Delta A^* = \Delta A/H^2,\ \xi = t\Psi,\ n = H/t, \quad \text{...} \quad (96)$$

which, when substituted into (95), yield

$$\Delta w_0(\rho) = -\frac{s_m n^2 \Delta A^*}{2\pi}\int_0^{\infty} \xi\, \kappa_2^n J_0(n\rho\xi)\,\mathrm{d}\xi, \quad \text{........} \quad (97)$$

where

$$a = \frac{\xi^4}{24(1-\nu^2)} + 1,$$

$$\kappa_2 = \frac{1}{2}\left[(\beta+1)a - \sqrt{(\beta+1)^2 a^2 - 4\beta}\right]. \quad \text{...............} \quad (98)$$

Next, the quasi-continuum model is examined. On the basis of (88) and (89), the influence function is defined as

$$W_0(r,0) = -\frac{1}{2\pi}\int_0^{\infty} \Psi\, e^{-\gamma_1 n} J_0(r\Psi)\,\mathrm{d}\Psi. \quad \text{...........} \quad (99)$$

The expression of the elementary displacement is now given by

$$\Delta w_0(\rho) = -\frac{s_m n^2 \Delta A^*}{2\pi}\int_0^{\infty} \xi\, e^{-\gamma_1 n} J_0(n\rho\xi)\,\mathrm{d}\xi, \quad \text{...} \quad (100)$$

where use was made of the dimensionless quantities in (96). In the above integrand,

$$\gamma_1 = \frac{1}{2}\left[\sqrt{(\beta-1)^2 + \frac{(\beta+1)\xi^4}{6(1-\nu^2)}} - (\beta-1)\right] \ge 0, \quad (101)$$

which is a non-negative quantity.

Frequently, the influence function is defined in a slightly different form in subsidence engineering[2,18]. According to this definition, the elementary subsidence is given by

$$\Delta w_0(\rho) = -\frac{s_m}{\pi} f(\rho^2)\,\Delta A^*. \quad \text{..........................} \quad (102)$$

This formulation arose from the earlier discussed hypothesis requiring that the extraction of a seam of constant thickness, when mined over an infinite area, leads to uniform subsidence, the magnitude of which equals the roof and floor convergence. As was seen earlier, in a uniformly stratified mass this hypothesis is fulfilled if $\beta \ge 1$. It is simple to show that the requirement of the hypothesis is satisfied in a more general sense if, and only if,

$$\int_0^{\infty} f(t)\,\mathrm{d}t = 1. \quad \text{.......................................} \quad (103)$$

The two versions of the influence function are related in a simple manner:

$$W(\rho) = -\frac{1}{\pi H^2} f(\rho^2). \quad \text{...............................} \quad (104)$$

The distinct layer and quasi-continuum versions of the influence function follow from the results in (95) and (100) respectively, and from the definition in (102). Thus, the influence function in the case of distinct layers becomes

$$f(\rho^2) = \frac{n^2}{2}\int_0^{\infty} \xi\, \kappa_2^n J_0(n\rho\xi)\,\mathrm{d}\xi \quad \text{.......................} \quad (105)$$

and, for the quasi-continuum, is given by

$$f(\rho^2) = \frac{n^2}{2}\int_0^{\infty} \xi\, e^{-n\gamma_1} J_0(n\rho\xi)\,\mathrm{d}\xi. \quad \text{..................} \quad (106)$$

It should be noted that, for $\beta = 1$, (101) yields $\gamma_1 = \xi^2/2\sqrt{3(1-\nu^2)}$ and, in this case, (106) can be evaluated in closed form:

$$f(\rho^2) = \frac{1}{2} n\sqrt{3(1-\nu^2)}\, e^{-\sqrt{3(1-\nu^2)}\, n\rho^2/2}, \quad \text{................} \quad (107)$$

which is the same as the result obtained earlier[1,2,18]. Also, this Gaussian function is identical in essence to the influence function proposed by Knothe[19] and used by others[3,4] with some modifications since.

The formulae in (105) and (106) are identical apart from the difference between functions κ_2 and $\exp(-\gamma_1)$. The same observation is true for the expressions in (95) and (100). Simple numerical work shows that functions

κ_2 and $\exp(-\gamma_1)$ are surprisingly close to each other for values of β that are likely to be relevant in practice. Assume that the number of effective layers in the roof strata, n, ranges from 10 to 20. This appears to be a realistic range. In Table I the ratios of the Young's modulus of the immediate floor layer, E_n, to that of the surface layer, E_o, are given for some values of β. The tabulation indicates that practical values of β are unlikely to exceed 1,1 in most cases.

TABLE I

THE RATIO E_n/E_0 AS A FUNCTION OF β AND n

n	10	12	14	16	18	20
1,00	1,00	1,00	1,00	1,00	1,00	1,00
1,05	1,63	1,80	1,98	2,18	2,41	2,65
1,10	2,59	3,14	3,80	4,59	5,56	6,73
1,15	4,05	5,35	7,08	9,36	12,38	16,37
1,20	6,19	8,92	12,84	18,49	26,62	38,34

Table II gives values of the subsidence influence functions for models involving distinct layers and the equivalent quasi-continuum. These values were computed by numerical integration for the model consisting of distinct laminae from the expression in (105) and for the quasi-continuum model from (106) respectively. This tabulation supports two deductions. It shows that the difference between the two types of influence functions is negligible. Thus, it seems permissible to employ the version of the models that offers some advantage in the solution to any particular problem.

The tabulation reveals also that there are fundamental differences between functions $f(\rho^2)$ corresponding to $\beta = 1$ and those for which $\beta \neq 1$. The values are calculated for $\beta = 0,9,\ 1,0,\ 1,1$. If $\beta = 1$, functions $f(\rho^2)$ decrease monotonically and approach zero asymptotically as ρ increases to infinity. If $\beta > 1$, $f(\rho^2)$ are *not* monotonic but, as ρ increases, the functions oscillate between positive and negative values as they approach zero. The common feature of all functions for which $\beta \geq 1$ is that

TABLE II

COMPARISON OF SUBSIDENCE INFLUENCE FUNCTIONS CORRESPONDING TO MODELS CONSISTING OF DISTINCT LAYERS, (105), AND EQUIVALENT CONTINUUM, (106)

ρ		0	0,1	0,2	0,3	0,4	0,5	0,6
β = 1,1	From (105)	+ 20,5774	+ 17,5446	+ 10,7189	+ 4,3772	+ 0,8145	− 0,3598	− 0,4313
	From (106)	+ 20,2389	+ 17,3844	+ 10,7129	+ 4,4359	+ 0,8482	− 0,3613	− 0,4430
β = 1,0	From (105)	+ 13,5897	+ 11,8601	+ 7,8858	+ 3,9978	+ 1,5465	+ 0,4565	+ 0,1026
	From (106)	+ 13,5765	+ 11,8529	+ 7,8875	+ 4,0006	+ 1,5467	+ 0,4558	+ 0,1024
β = 0,9	From (105)	+ 3,9195	+ 3,3324	+ 2,0153	+ 0,8009	+ 0,1305	− 0,0800	− 0,0840
	From (106)	+ 4,3150	+ 3,6602	+ 2,1977	+ 0,8617	+ 0,1353	− 0,0870	− 0,0890

ρ		0,7	0,8	0,9	1,0	1,1	1,2	1,3
β = 1,1	From (105)	− 0,2500	− 0,1066	− 0,0335	− 0,0040	+ 0,0048	+ 0,0055	+ 0,0039
	From (106)	− 0,2573	− 0,1086	− 0,0329	− 0,0030	+ 0,0055	+ 0,0059	+ 0,0040
β = 1,0	From (105)	+ 0,0175	+ 0,0022	+ 0,0002	+ $1{,}4.10^{-5}$	+ $5{,}5.10^{-7}$	+ $5{,}7.10^{-8}$	
	From (106)	+ 0,0175	+ 0,0023	+ 0,0002	+ $1{,}7.10^{-5}$	+ $1{,}0.10^{-6}$	+ $4{,}4.10^{-8}$	+ $1{,}5.10^{-9}$
β = 0,9	From (105)	− 0,0453	− 0,0174	− 0,0043	+ 0,0003	+ 0,0013	+ 0,0011	+ 0,0007
	From (106)	− 0,0478	− 0,0187	− 0,0049	+ 0,0001	+ 0,0013	+ 0,0012	+ 0,0007

N = 16, ν = 0,2

they satisfy (103). For $\beta < 1$, $f(\rho^2)$ have a similar oscillatory nature but, as can be deduced on the basis of the earlier discussion, instead of satisfying (103) they yield

$$_0\int^{\infty} f(t)\,dt < 1 \quad \beta < 1. \qquad (108)$$

No influence functions with oscillatory features appear to have been proposed before. It seems that heterogeneity in physical properties can have far-reaching consequences. For example, it may lead to uplift of the ground surface at points that fall outside the projection of the mined-out area.

Such uplifts have been observed by numerous investigators in the field, but no attempt seems to have been made to provide a mechanistic explanation of the phenomenon.

Homogeneous Quasi-continuum

The deliberations in this paper so far have devoted no attention as to how the convergence distribution, $s(x,y)$, in the integrand of (25) is to be determined. This is a serious shortcoming since the convergence distribution is usually defined by a set of simultaneous integral equations[15] that are often very difficult to solve. Fortunately, the homogeneous version of the quasi-continuum model, that is the uniformly stratified model in which all layers have identical Young's moduli ($\beta = 1$), offers an opportunity for a much simpler solution.

Fundamental Solution

The differential equation in (18) reduces to

$$\lambda^2\nabla^4 w - \frac{\partial^2 w}{\partial z^2} = 0, \qquad (109)$$

if the ground is homogeneous; that is, when $\beta = 1$. Also, on the basis of (22) and (23), the vertical stress is given by

$$\sigma = E\frac{\partial w}{\partial z}, \qquad (110)$$

where, for the sake of simplicity, constant E_0 is replaced by E. The integral transform of (109) is

$$\lambda^2\Psi^4\bar{w} - \frac{d^2\bar{w}}{dz^2} = 0, \qquad (111)$$

regardless of whether the Hankel, double Fourier, or Fourier cosine transform is employed. The general solution to this equation is

$$\bar{w} = C_1(\Psi)e^{\lambda z\Psi^2} + C_2(\Psi)e^{-\lambda z\Psi^2}, \qquad (112)$$

which, of course, could have been obtained directly from (82) through the substitution of $\beta = 1$ into (83).

The unknown C_i functions in the solutions are determined from the boundary conditions. Here this will be done in two steps. First, the seam is assumed to be in an infinite medium with the origin of the x,y,z_1 coordinate system located in the seam (Fig. 2). Next, a second seam, a fictitious 'mirror image' seam, is placed $2H$ distance above the actual seam. The origin of the x,y,z_2 coordinate system is at this horizon. This mirror-image seam is also taken to be in an infinite medium. It is assumed that the excavations in the two seams are identical but, while convergence occurs in the actual seam, divergence takes place in the mirror-image seam. The distributions of the convergence and the divergence are identical in magnitude but opposite in sign. Consequently, the sum of the two solutions provides a stress-free surface halfway between the seams; that is, at the ground surface. Thus, this union of solutions is the solution corresponding to the actual excavation at finite depth. In the following sections, components corresponding to the actual and to the mirror-image seams will have subscripts a and m respectively.

The solution for the actual seam at infinite depth follows from (112).

In the roof this takes the form:

$$\bar{w}_a^{(-)} = C(\Psi)e^{\lambda z_1\Psi^2}, \qquad \bar{\sigma}_a^{(-)} = E\lambda\Psi^2 C(\Psi)e^{\lambda z_1\Psi^2}, \qquad (113)$$

which in the floor becomes

$$\bar{w}_a^{(+)} = -C(\Psi)e^{-\lambda z_1\Psi^2}, \quad \bar{\sigma}_a^{(+)} = E\lambda\Psi^2 C(\Psi)e^{-\lambda z_1\Psi^2}. \qquad (114)$$

In the case of the mirror-image seam, only the results corresponding to the floor are of interest:

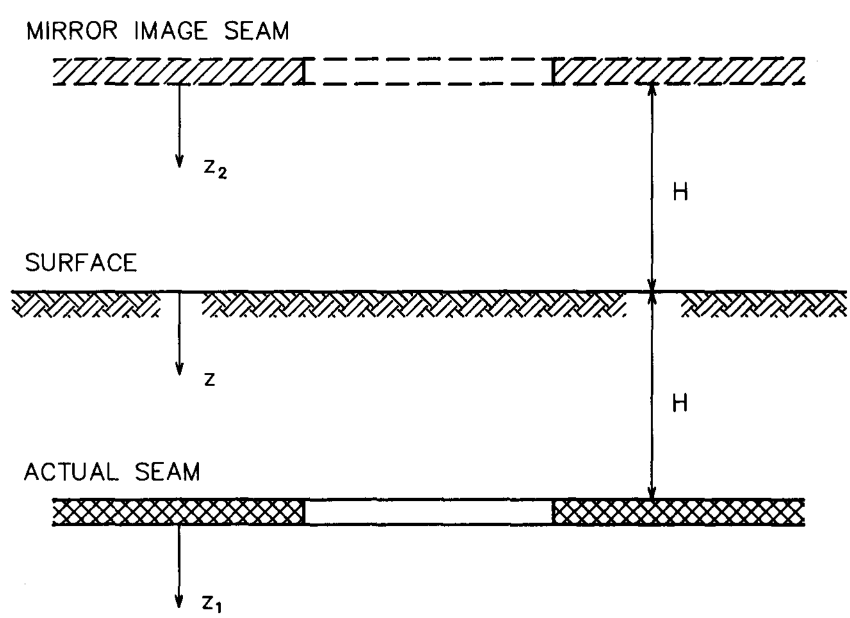

Fig. 2—Finite depth solution:the concept of mirror-image seam

$$\bar{w}_m^{(+)} = C(\Psi)e^{-\lambda z_2 \Psi^2},\ \bar{\sigma}_m^{(+)} = -E\lambda\Psi^2 C(\Psi)e^{-\lambda z_2 \Psi^2}. \quad (115)$$

Now superimpose these results on those corresponding to the actual excavations. The resultant components are given next.

In the *roof* of the seam these are

$$\bar{w}^{(-)} = C(\Psi)(e^{\lambda z_1 \Psi^2} + e^{-\lambda z_2 \Psi^2}),$$
$$\bar{\sigma}^{(-)} = E\lambda\Psi^2 C(\Psi)(e^{\lambda z_1 \Psi^2} - e^{-\lambda z_2 \Psi^2}), \quad (116)$$

and *beneath* the seam they become

$$\bar{w}^{(+)} = C(\Psi)(-e^{-\lambda z_1 \Psi^2} + e^{-\lambda z_2 \Psi^2}),$$
$$\bar{\sigma}^{(+)} = E\lambda\Psi^2 C(\Psi)(e^{-\lambda z_1 \Psi^2} - e^{-\lambda z_2 \Psi^2}). \quad (117)$$

Now observe that

$$-\Psi^2\bar{w}^{(-)} + \frac{1}{E\lambda}\bar{\sigma}^{(-)} = -2\Psi^2 C(\Psi)e^{-\lambda z_2 \Psi^2} = \frac{2}{\lambda E}\bar{\sigma}_m^{(+)}, \quad (118)$$

and that

$$-\Psi^2\bar{w}^{(+)} - \frac{1}{E\lambda}\bar{\sigma}^{(+)} = 0. \quad (119)$$

As the integral transform of $\nabla^2 w$ is $-\Psi^2\bar{w}$ with respect to all three transforms used in this paper, the results in (118) and (119) can be put in the following forms:

$$\nabla^2 w^{(-)} = -\frac{1}{E\lambda}(\sigma^{(-)} - 2\sigma_m^{(+)}), \quad (120)$$

$$\nabla^2 w^{(+)} = \frac{1}{E\lambda}\sigma^{(+)}. \quad (121)$$

Since, at $z = H$, $s = w^{(+)} - w^{(-)}$ and $\sigma^{(+)} = \sigma^{(-)}$, subtract (120) from (121) to obtain

$$\nabla^2 s = \frac{2}{\lambda E}(\sigma - \sigma_m^{(+)}). \quad (122)$$

This is the equation that defines the convergence distribution on the seam horizon[2,20]. If the seam is at great depth, $\sigma_m^{(+)}$ can be neglected and the convergence distribution is defined by a second-order differential equation, which reduces to a Poisson's equation if the excavation is unsupported. This is so because in this case the right-hand side of the equation is independent of s. In general, however, both σ and $\sigma_m^{(+)}$ are functions of the convergence. It should be emphasized that the stress components in (122) are both *induced* vertical stresses.

It is, perhaps, of some interest that the equation in (121) is mathematically analogous to the simplest version of the 'stochastic' medium of Litwiniszyn and his co-workers. If, from (110), $\sigma = E\partial w/\partial w$ is substituted into (121), the latter can be put in the following form:

$$\frac{\partial w^{(+)}}{\partial z} = \lambda\,\nabla^2 w^{(+)}, \quad (123)$$

which is identical to the equation controlling the behaviour of the homogeneous incompressible stochastic medium[21].

Convergence Distribution

The seam horizon may contain several regions where the coal has been extracted and the roof is supported by various means. Each of these regions can be surrounded by a closed contour. Some of these regions are in contact with one another, and the encircling contours butt against one another. In principle, the solutions corresponding to all regions are interdependent. Consequently, the convergence distribution in an area cannot be derived independently from the conditions prevailing in other regions. (It will be seen that there is an exception to this statement.)

Physical considerations demand that the convergence function $s(x,y)$ should be continuous throughout the plane. Here it will be advantageous to accept Sokolnikoff's classification of the continuity of functions. He defines that a function $u(x,y,z)$ is of class C^k if u itself with its first, second, and up to the k-th derivatives are all continuous[22]. In terms of this definition, function $s(x,y)$ is, at least, of class C°.

It is axiomatic that the resultant of the vertical forces induced by horizontal tabular excavations on any horizontal plane, including the plane of the seam, must be zero. To investigate the consequences of this criterion in the present case, assume that all mined areas are bound externally by a contour D that encloses an area A (Fig. 3). It follows from (122) and (115) that the resultant of the induced vertical stress σ over area A is

$$F = {}_A\!\int \sigma\,\mathrm{d}A = \lambda E\,({}_A\!\int \nabla^2 w_m^{(+)}\,\mathrm{d}A + \frac{1}{2}\,{}_A\!\int \nabla^2 s\,\mathrm{d}A). \quad (124)$$

Care must be taken in evaluating this integral. Although, on the seam horizon, $w_m^{(+)}$ is a class C^∞ function, the same cannot be said about the convergence distribution $s(x,y)$. Contour D_j encloses the mined-out area A_j (Fig. 3). It follows from Green's theorem in a plane[23] that, over the union of extracted regions,

$$\sum_{j=1}^{N} {}_{A_j}\!\int \nabla^2 s\,\mathrm{d}A = \sum_{j=1}^{N} {}_{D_j}\!\int (s_x^- n_{jx} + s_y^- n_{jy})\,\mathrm{d}t, \quad (125)$$

where N is the number of mined-out areas, n_j is the exterior normal to contour D_j with rectangular components:

$$n_{jx} = \cos(x, n_j),\quad n_{jy} = \cos(y, n_j). \quad (126)$$

Also $s_x^- = \partial s/\partial x$ and $s_y^- = \partial s/\partial y$. Here the negative sign over s_x and s_y signifies that the derivatives are to be evaluated on the *interior* side of contour D_j. Let the unmined area within contour D be A_o, that is

$$A = \sum_{j=0}^{N} A_j.$$

Area A_o is the multiply connected domain that is bounded externally by D and internally by contours D_j. In this case the Green's theorem yields[23]

$${}_{A_o}\!\int \nabla^2 s\,\mathrm{d}A = {}_D\!\int (s_x^- n_x + s_y^- n_y)\,\mathrm{d}t - \sum_{j=1}^{N} (s_x^+ n_{jx} + s_y^+ n_{jy})\,\mathrm{d}t, \quad (127)$$

where s_x^+ and s_y^+ are to be evaluated on the *exterior* sides of contours D_j.

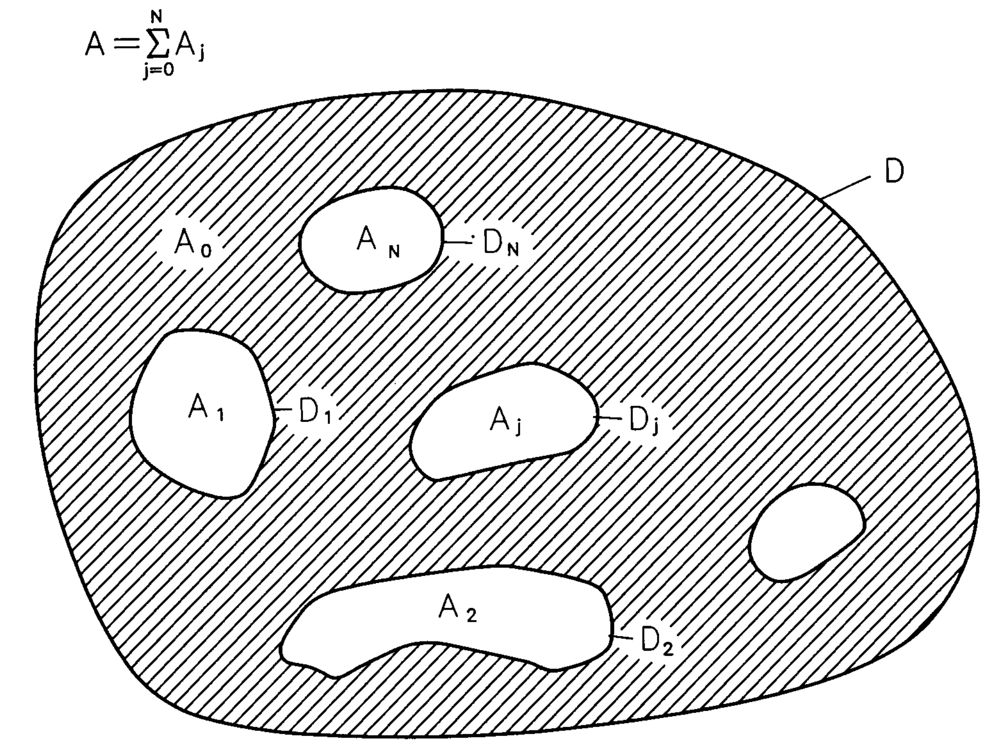

Fig. 3—Conceptual diagram showing mined-out areas, A_j, enclosed by contours D_j (j = 1, 2, 3, ... N) and embedded in the unmined region A_0 which, in turn, is surrounded by contour D

Now the resultant force F in (124) can be expressed in terms of the following integrals:

$$F = \lambda E \{_D\int (w_{mx}^{(+)-} n_x + w_{my}^{(+)-} n_y)\, dt + \frac{1}{2}\, _D\int (s_x^- n_x + s_y^- n_y)\, dt - \frac{1}{2}\sum_{j=1}^{N}\, _{D_j}\int [(s_x^+ - s_x^-) n_{jx} + (s_y^+ - s_y^-) n_{jy}]\, dt\}. \quad (128)$$

To obtain this result, advantage was taken of the continuity of the first derivatives of $w_m^{(+)}$ across contours D_j. In view of the form of the last term in (128), this continuity ensures that the contribution of the integral of $\nabla^2 w_m^{(+)}$ is restricted to that arising from contour D.

Now, if A becomes the whole xy plane, then

$$F = -\frac{1}{2}\lambda E \sum_{j=1}^{N}\, _{D_j}\int [(s_x^+ - s_x^-) n_{jx} + (s_y^+ - s_y^-) n_{jy}]\, dt, \quad (129)$$

because the first derivatives of both s and $w_m^{(+)}$ approach zero at infinity.

In general, the equilibrium of forces can be put in the following form:

$$F + P = 0, \quad (130)$$

where F is the resultant of the induced vertical stress σ, and P is the resultant of possible concentrated forces. It is necessary to discriminate two situations. The first case is when $F = 0$ and there are no concentrated forces. This is the more familiar situation in stress analysis. The second case is when $F \neq 0$ and concentrated forces do exist to ensure the equilibrium of the induced forces.

Case $F = 0$

Earlier-mentioned physical conditions of continuity require that $s(x,y)$ is at least a class C° function. This requirement ensures that the first directional derivative *along* any contour D_j is continuous when moving across the contour at any point. Note that $(s_x n_{jx} + s_y n_{jy})$ is the first directional derivative in a direction *normal* to contour D_j. The result in (129) indicates that $F = 0$ if this normal derivative is continuous when moving across the contours.

Thus, the *sufficient* and *necessary* condition to ensure that (i) the induced vertical forces are in equilibrium on the xy plane and that (ii) there are no concentrated forces acting in the system, is that function $s(x,y)$ must be at least of class C^1.

Case $F \neq 0$

Some concentrated forces must exist on the seam horizon in this case, and the resultant of these forces is given by

$$P = -F = \frac{1}{2}\lambda E \sum_{j=1}^{N}\, _{D_j}\int [(s_x^+ - s_x^-) n_{jx} + (s_y^+ - s_y^-) n_{jy}]\, dt. \quad (131)$$

Thus, a sufficient condition to ensure the equilibrium of the induced vertical forces when $s(x,y)$ is of class C°, but *not* of class C^1, is that along contour D_j there is a concentrated line force of intensity

$$p_j = \frac{1}{2}\lambda E [(s_x^+ - s_x^-) n_{jx} + (s_y^+ - s_y^-) n_{jy}]. \quad (132)$$

Obviously, this line force intensity satisfies (131); that is,

$$P = \sum_{j=1}^{N}\, _{D_j}\int p_j\, dt. \quad (133)$$

The simplest, non-trivial examples of the application of the results in this section involve a single parallel-sided long panel of span $2L$. Assume that the long axis of the panel is in the y direction and that the roof is unsupported. Therefore, the induced stress within the panel equals the virgin vertical stress, q, with a negative sign. Assume that the span is small in relation to the depth; therefore, $\sigma_m^{(+)}$ can be ignored. In this case, according to (122), the convergence in the panel is controlled by

$$\frac{d^2 s_p}{dx^2} = -\frac{2q}{\lambda E} \quad |x| \leq L, \qquad (134)$$

where the variation in s_p along the y-axis is regarded as negligible. The solution of (134) is

$$s_p(x) = \frac{q}{\lambda E}(-x^2 + C_1 x + C_2). \qquad (135)$$

Two versions of the solution are obtained. First, it will be assumed that there is a linear relationship between the vertical induced stress acting on the unmined coal outside the panel and the compaction of the seam; that is,

$$\sigma = \mu\, s_r. \qquad (136)$$

Here, if the coal obeys the isotropic Hooke's law with E_s and ν_s as its Young's modulus and Poisson's ratio, the factor of proportionality is $\mu = E_s/(1-\nu_s^2)M$ if the point in question is close to the ribside ($\sigma_x = 0$, $\epsilon_y = 0$), or $\mu = (1-\nu_2)E_s/(1-\nu_s-2\nu_s^2)M$ for points remote from the ribside ($\epsilon_x = \epsilon_y = 0$), where M is the seam thickness. These assumptions, after using (122), lead to the differential equation

$$\frac{d^2 s_r}{dx^2} - \frac{2\mu s_r}{\lambda E} = 0, \qquad (137)$$

the solution of which is

$$s_r(x) = C_3 e^{a(x-L)} + C_4 e^{-a(x-L)}, \qquad (138)$$

where

$$a^2 = \frac{2\mu}{\lambda E}. \qquad (139)$$

To ensure the symmetry of the convergence distribution, it is necessary to take $C_1 = 0$ in (135). Also, to avoid the unbounded increase in $s_r(x)$ for large positive values of x, it is necessary to have $C_3 = 0$ for $x \geq L$. The conditions arising from the continuity in convergence and its slope provide the remaining two constants and lead to the solution

$$s_p(x) = \frac{q}{\lambda E}\left[\frac{L}{a}(2 + aL) - x^2\right],$$

$$s_r(x) = \frac{2qL}{a\lambda E}e^{-a(x-L)}. \qquad (140)$$

The convergence distribution on the left-hand side of the panel (i.e. $x \leq -L$) can be obtained from s_r by changing the sign of the exponent to positive. It is simple to show that this solution ensures the equilibrium of induced forces on the seam horizon and that there are no concentrated forces acting on this plane.

The second solution is a simplification of the first. It is assumed that the compression of the seam is negligible and can be ignored. This solution can be obtained from (140) by letting $a \to \infty$:

$$s_p(x) = \frac{q}{\lambda E}(L^2 - x^2), \quad s_r(x) = 0. \qquad (141)$$

These results suggest that no induced stress acts on the unmined seam. This follows from (122) since it is assumed here that $\sigma_m^{(+)} = 0$. Hence, the equilibrium can be maintained only through the presence of concentrated forces. This is, in fact, the case. A line of compressive concentrated force of intensity qL per unit length runs along both edges of the panel, compensating for the induced tensile force acting inside the panel. The magnitude of this latter force is $-2qL$ per unit length.

This feature of the result in (141) is typical for all those solutions that ignore the compression of the unmined seam. This observation shows both the strength and the weakness of the model. Its simplicity permits the derivation of solutions with relative ease. If $\sigma_m^{(+)}$ can be ignored, then (122) can be solved independently for each isolated excavation in the plane of the seam. However, care must be exercised to ensure that the solutions so deduced do not represent unacceptable over-simplifications of reality.

Conclusions

A frictionless laminated model has been revived in this paper. Previously published accounts of the model were restricted to the discussion of quasi-continua, where the individuality of layers is blurred. Here the treatment is reasonably comprehensive. It covers the case where the model consists of distinct layers, the laminae being either arranged entirely heterogeneously or the stratification being systematic in some sense. The quasi-continuum versions are also discussed in reasonable depth. It is shown that, when a distinct layer model can be approximated with its continuous version, the deviation between solutions is small provided the number of layers between the seam and the point of comparison is more than about ten.

Perhaps the most valuable result is the equation controlling the convergence distribution in the homogeneous quasi-continuum. This equation lends itself to a relatively simple solution of an otherwise complex problem.

The purpose of the present paper is to present the foundation for the solution of various practical problems. The intention is to follow this presentation with a discussion of a variety of applications, including the prediction of surface subsidence and pillar load, computation of the stiffness of a rock mass, discussion of pillar strength on the basis of the confined core concept of Wilson[24], and so on.

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New information products for the minerals industry*

Mintek, one of the world's leading mineral-research organizations, has announced a unique computerized information-retrieval service that has been developed especially for the mineral industry. Mintek's two in-house databases—MINTEXT and MINCAT—are relatively small and highly specialized, and contain only entries that have been carefully selected for their relevance, thus ensuring that the information retrieved has an inherently high value. By agreement with the UK Institution of Mining and Metallurgy, the IMMAGE database, the world's foremost mineral-industry database, is also available to South African users.

The subjects covered by these databases include mineralogy, mining technology, extractive metallurgy, ore-dressing, mineral processing, analytical procedures, extraction technology, hydrometallurgy, pyrometallurgy, process chemistry, physical metallurgy, economic geology, mineral economics, and process control. The different emphases of the three databases ensure that the results of searches are complementary.

Information searches can be carried out by Mintek's own experienced information staff, or passwords can be supplied so that users can undertake their own literature surveys, either at Mintek or by remote access via data-communication networks. The databases are also available on floppy disks for installation on subscribers' computers. Comprehensive user manuals and training courses are available from Mintek.

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New master's degree in mineral economics*

The Department of Mining Engineering at the University of the Witwatersrand has introduced a new master of science degree in the application of business and economic principles to the mining and mineral industries. The new degree, which is offered from 1991, will be the only degree in the country to provide a qualification in mineral economics. It is aimed at professionals such as engineers, geologists, and economists operating in the fields of mining, investment, or the public service.

'Exposure to mineral economics has become crucial to decision-makers in the demanding fields of mineral extraction and beneficiation', says Prof. Huw Phillips, Head of the Department of Mining Engineering. 'Mineral economics is an interdisciplinary science that fuses the principles of economics with those of engineering and the earth sciences. It aims to qualify candidates both in the fundamental principles of mineral economics and in specialist techniques such as financial valuation and geostatistics.'

The mineral economics programme offers a choice of 16 courses, of which 12 must be successfully completed for the MSc degree and six to obtain a Graduate Diploma. Part-time candidates will have to complete at least three courses in any year of study. Four of the courses (mineral economics, mineral policy and investment, minerals marketing, and decision-making for mining investments) are compulsory for the MSc qualification.

Each course will be presented over two weeks of full-time lectures. Professionals who do not wish to obtain a qualification can attend courses of particular interest to them.

The Department of Mining Engineering is currently taking registrations for next year's courses. Enquirers should telephone Mr Anton Von Below at (011) 716-5192 with any queries.

* Released by Lynne Hancock Communications, P.O. Box 1564, Parklands 2121.

Members of academic staff in the Department of Mining Engineering who are involved in the new Mineral Economics Programme include (from left) Lecturer Mr Anton Von Below; Senior Lecturer Dr Isobel Clark, who heads the programme; head of the Department of Mining Engineering, Prof. Huw Phillips; honorary professorial research fellow, Prof. Danie Krige; and Senior Lecturer Mr Rudi Sesink Clee