Mathematical Models of Orebodies

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SYNOPSIS

The reasons for constructing a model of an orebody for which both the parent population and the sampling distribution are known are discussed and details of the model are given. A brief summary of Matheron's theory of regionalized variables is included, with particular reference to his kriging estimators and their variances. Using the model, a comparison is then made between these kriging estimators and other commonly-used estimators. These are compared to find the one with the minimum variance, and an analysis is included to investigate whether any of the estimators is biased. Also included is a comparison between theoretical and experimental variances for some of the estimators.

INTRODUCTION

It is often stated that the grade and tonnage of an orebody are not known until it has been mined out. Often nothing could be further from the truth. Ore left in stopes, dilution and sampling difficulties, etc., can all contribute to producing an imperfect estimate of the original ore resources. These sources of error will impair comparison between estimated and mined grades and tonnages, and will also tend to conceal any deficiencies in mining and milling. Various methods exist for evaluating grade and tonnage, depending to a certain extent on the data available, but any direct comparison between the methods is difficult because of the sources of error mentioned above. It is important that a confidence interval be associated with every estimate of grade and tonnage. Some of the evaluation methods have a theoretical confidence interval but it becomes very difficult to test these in practice.

This is the field in which a model can be of use. If models of geostatistical structures can be constructed which display features comparable with real situations then evaluation techniques can be tested and compared. Also, discrepancies can be traced to their sources.

Of particular interest are Matheron's kriging estimators based on his theory of regionalized variables (Matheron, 1962, 1963, 1965). Koch, *et al* (1971) state that 'Unfortunately, few, if any, studies comparing evaluations by Matheron's methods with those by other methods have yet been published.' It is hoped that this paper will begin the process of filling this gap.

THE MODEL

A description of some aspects of an actual orebody can be based only on an appraisal of the samples cut from it. A parent population can give rise to an infinite number of sampling sets and, conversely, there is no reason to suppose that a given sample set might not have been drawn from any one of several different parent populations. In the model it is essential to construct a parent orebody which is known completely. Samples can then be taken from the parent orebody in order to obtain the sample distribution. This image of the parent orebody produced by the samples can then be compared directly with the parent orebody itself, so that estimation methods and confidence limits can be tested.

To define an orebody in mathematical terms it is necessary to decide on the relevant parameters. These appear to be:

(i) *The statistical distribution of the samples.* This distribution is obtained by random sampling over a given region but without regard to the relative positions of the samples within the orebody. The statistical distribution may be normal, binomial, lognormal, etc., but the three-parameter lognormal distribution appears to be the most common, especially in gold mines (Krige, 1960). For this reason it was decided to consider only sampling models which conformed to lognormal distributions.

(ii) *The geometrical, or spatial, distribution of the assays.* Generally, neighbouring sample values are correlated. Techniques used to describe the spatial distribution include trend surface analysis, deterministic methods and geostatistical methods based on the theory of regionalized variables. (Matheron, 1962, 1963, 1965). It is this method which has been used in the model, and the relevant features of it are discussed in the next section.

GEOSTATISTICS

Every evaluation technique is based on extending data from individual samples to a volume, or panel, of ore. The difference between techniques lies in the different methods used in allocating weighting factors to the samples. Conventional statistical methods were employed until 1951 when Krige improved the estimator by taking into account the relationship between the sample and the panel to which its value was extended. In 1962, G. Matheron found results similar to those of Krige by using his theory of regionalized variables. An assay value $f(x)$, expressed, for example, in in.-dwt units, is a regionalized variable, as it is a function of:

(i) its position, x, in the mineralized volume,

(ii) the geometry of the sample, and

(iii) the orientation of the sample.

THE VARIOGRAM

The basic diagnostic tool in the theory of regionalized variables is the intrinsic semi-variogram, which is given by:

$$\gamma(h) = \tfrac{1}{2} E_v \{[f(x) - f(x+h)]^2\}$$

where v is the field in which $f(x)$ is defined and h is a variable vector. In practice, only a finite number of samples are taken and so the value of the experimental semi-variogram, $\gamma(h)$, at lag h is given by:

$$\gamma(h) = \frac{\frac{1}{2}\Sigma[f(x+h) - f(x)]^2}{L - h}$$

where L is the total number of assays in the length sampled.

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The shape of the variogram depends on the characteristics of the regionalized variable. The continuity and regularity of the regionalized variable are shown by the behaviour of $\gamma(h)$ near the origin, and details of different forms of variogram are described in Blais, *et al* (1968). One type of variogram, the spherical type, is described briefly below.

SPHERICAL VARIOGRAM

This is an intrinsic variogram characterizing a transition phenomenon, that is, one in which $\gamma(h)$ reaches a finite value as h increases indefinitely. The value of h at which $\gamma(h)$ reaches a finite value is called the range, and is denoted by a. The variogram has the form:

$$\gamma(h) = C_o + \tfrac{1}{2}C[3(h/a) - (h/a)^3] \text{ for } h < a$$
$$= C_o + C \qquad \text{for } h \geqslant a$$

and is shown in Fig. 1.

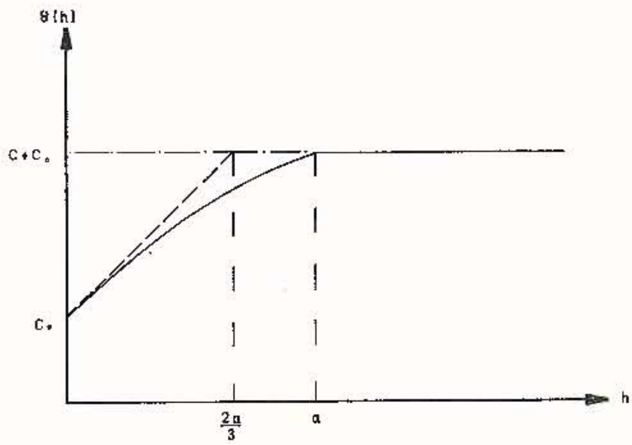

Fig. 1. The spherical variogram.

Here C_o is the nugget variance, and $\varepsilon = C_o/C$ is the nugget effect.

Kriging

Here, the value of a panel of ore is estimated by assigning weighting factors to the available samples. The weights depend on the geostatistical parameters of the deposit, calculated from the variogram, and on the geometry of the assays relative to the panel being kriged. Different kriging procedures are available depending on the panel size, the position of the assays relative to each other and to the panel, and the nugget effect. There are two classes of kriging, namely, random and regular. Regular kriging procedures can be used when assays have been taken on a regular grid. Random kriging procedures can be used when the data conform to a random stratified grid, that is, when a rectangular grid can be fitted to the data so that one assay is placed at random in each grid rectangle.

With each kriged estimate is associated a variance which can be used to calculate confidence limits for the estimator.

Two kriging procedures are discussed here in more detail.

Small panels

This procedure can be applied to random stratified grids, and to regular grids when $\varepsilon > 0.3$. The kriged estimator, t, of a panel has the form:

$$t = \lambda x + \mu y + (1 - \lambda - \mu) z,$$

where x is the assay of the panel being estimated, y is the mean of x and the eight assays in the eight surrounding panels, and z is the mean value of all the assays in the deposit. Parameters λ and μ are the appropriate kriging coefficients, the equations for which are given in Serra (1967).

If there is no sample in the panel being estimated, this panel can still be kriged using the formula

$$t = \mu y + (1 - \mu) z$$

where y is the mean of the samples in the eight adjacent panels, and μ and the kriging variance are again given in Serra (1967).

Large panels

A large panel may contain nine or more samples and the kriged estimator of the panel takes the form

$$t = (1 - \lambda) y + \lambda z$$

where y is the mean of the samples in the panel and z the mean of all the samples in the deposit. These kriging equations are again given in Serra (1967).

THE PARENT OREBODY

The number of blocks in the model is limited to some extent by the capacity and availability of the computer. Leeds University has an English Electric K.D.F.9 computer run on a time-sharing basis. In order to get a satisfactory turn-round time it was necessary for the program to occupy less than 15k byte of store. On the basis of this it was decided to construct a two-dimensional square model orebody and to divide it into 42 × 42 blocks, each block being allocated a mean assay value.

To produce correlation between neighbouring blocks, the mean value of a particular block was obtained by placing a square grid, each square representing one block, over a smooth surface represented by a trigonometrical expression. The mean value of a block was taken to be proportional to the height of the surface at the centre of each grid square.

The block values are obtained by using the expression

$$y_{jk} = 2C + \sum_{i=1}^{n} \{a_i[\sin(i\theta_j) + \sin(i\theta_k)] + b_i[\cos(i\theta_j) + \cos(i\theta_k)]\}.$$

The blocks are numbered relative to the top left hand corner of the orebody so that the j-th block in the k-th row is block (j, k). Hence,

$$\theta_j = j\alpha/42 \qquad j = 1, 2, \ldots 42,$$
$$\phi_k = k\alpha/42 \qquad k = 1, 2, \ldots 42,$$

where $\alpha < 3\pi/2$.

A surface of this form is isotropic in the sense that all sections through the orebody parallel to either of the edges have the same basic form differing only by a constant and a lateral shift.

The actual size of the orebody is not important as all measurements are made relative to the side of a block. Thus the model can represent either a complete orebody divided into 1 764 square mining blocks, or a section of an orebody divided into the same number of small panels.

THE SAMPLE SET

The method of obtaining a sample set from the parent orebody is to take each block in turn and draw a random sample from a statistical distribution with mean equal to the mean of the block, y_{jk}, and with a predetermined variance, w, common to all blocks. Such a random sample can be obtained with the aid of a pseudo-random number generator. Thus, this method will give a value for each block, and all 1 764 of these comprise the sample set from which the following are determined:

(i) the mean, variance, log-mean and log-variance,
(ii) the frequency and cumulative frequency distributions, and
(iii) an average variogram.

It may be noted that variograms are calculated for rows 1, 5, 9 . . . 41 and similarly for columns 1, 5, 9 . . . 41 so that the average variogram is the mean of 22 variograms.

ESTIMATION TECHNIQUES

Panels of ore having different sizes are estimated by different methods. In every case the panel is either a single block of ore or a combination of several complete blocks of ore. In the former case the true panel value is simply the block mean, and in the latter case the true panel value is the arithmetic mean of the block means which make up the large panel.

SINGLE PANELS

Where each block contains a sample two estimates are made, the sample value drawn from that block being one estimate and the kriging estimate the other.

By ignoring the sample in the panel being estimated a comparison can be made between the appropriate kriging estimator and an estimator based on an inverse square weighting method using the eight adjacent samples. The inverse square weighting estimator, t, of the panel a_0 is:

$$t = [(s_2 + s_4 + s_6 + s_8) + (s_1 + s_3 + s_5 + a_7)/2]/6$$

where $s_1 \ldots s_8$ are the samples from panels $a_1 \ldots a_8$. Since the model consists of 42^2 panels, 41^2 panels were estimated by these methods.

COMPOSITE PANELS

Large square panels of four different sizes were each estimated by three different methods. The panels estimated were of size 9, 16, 25 and 36 single panels. The estimation methods used were:

(i) the kriging estimator,
(ii) Sichel's 't' estimator (Sichel, 1952; Krige, 1960), and
(iii) the arithmetic mean of the sample set of the composite panel.

FORMAT OF RESULTS

For each panel, the deviation between each estimate and the true panel mean is found and the mean and standard deviation of these deviations are calculated. That is, if z_{ij} is the true value of panel (i, j) and y_{ij} is the estimate of z_{ij}, then the mean and variance of the deviations $(y_{ij} - z_{ij})$ of all the panels are calculated. This calculation is repeated using the various estimates.

The results in this form will determine the estimator with the minimum variance but they will not show if any of the estimators are subject to a regression effect, that is, whether they undervalue low-grade panels and over-value high-grade panels, or *vice versa*. To investigate this, the mean deviation between the estimated value and the true panel mean is checked for certain ranges of the true panel mean.

RESULTS

SMALL PANELS

A section of the results obtained is shown in Table I. These results are discussed below. It should be noted at this point that although a few of the model orebodies produced have nugget effects of less than 0.3, their panel values have nevertheless been estimated using random kriging procedures. For these panels the kriged estimate, therefore, will have a slightly higher variance than the estimate obtained by using a regular kriging procedure.

TABLE I

WITH SAMPLE IN ESTIMATED PANEL

Orebody No.	Theoretical s.d.	Kriging s.d.	Sample s.d.	Kriging mean deviation	Sample mean deviation	Regression slope of kriging	Regression slope of sample
1	79	75	215	6	7	−0.140	−0.048
2	64	61	163	−4	−6	−0.032	−0.011
3	42	35	98	−1	−1	0.003	0.023
4	83	76	236	3	3	−0.112	−0.017
5	64	51	156	−2	−1	−0.082	−0.005
6	106	83	225	−4	−4	−0.037	−0.002
7	72	58	160	−1	−1	−0.030	0.006
8	95	89	242	−4	−3	−0.070	0.002
9	57	48	144	2	2	−0.037	0.013
10	95	85	253	−3	−2	−0.104	−0.016

WITHOUT SAMPLE IN ESTIMATED PANEL

Orebody No.	Theoretical s.d.	Kriging s.d.	In. Sq. s.d.	Kriging mean deviation	In. Sq. mean deviation	Regression slope of kriging	Regression slope of In. Sq.	Nugget effect
1	83	79	84	6	6	−0.158	−0.078	0.94
2	69	63	65	−3	−3	−0.039	−0.045	0.10
3	45	35	37	0	0	−0.003	−0.005	0.21
4	88	78	87	3	3	−0.129	−0.048	0.92
5	67	55	58	−2	−2	−0.096	−0.033	0.69
6	112	87	89	−4	−4	−0.048	−0.048	0.21
7	77	60	62	−2	−2	−0.041	−0.041	0.24
8	100	93	98	−4	−3	−0.086	−0.052	0.53
9	60	50	54	2	2	−0.048	−0.019	0.48
10	100	89	96	−3	−2	−0.121	−0.044	0.88

With a sample in the panel being estimated

As would be expected the kriging estimate has a smaller standard deviation (s.d.) than the estimate made from the single sample taken within the panel. The s.d. of the kriging estimate varies from 35 to 89 in-dwt, whereas that for the single sample varies from 98 to 284 in-dwt. On average the latter s.d. is 2.8 times larger than that of the kriging estimate thus making the confidence limits for the kriging estimate only 0.36 times as wide as those for a single sample estimate.

The mean of the deviations, over a whole orebody, for both estimators varies between −14 and +10 in-dwt and there is never a difference of more than 2 in-dwt between the two mean deviations. The mean and s.d., over 56 model orebodies, of the mean deviations are −1 and 5, respectively, showing that on average the mean deviation of both estimators is not significantly different from zero.

A graph of the theoretical s.d. of the kriging estimate plotted against the experimental s.d. for small panels with a sample in the panel being estimated is shown in Fig. 2. All the points except one lie below the 45-degree line, showing that the theoretical s.d. was an over-estimate of that obtained in practice.

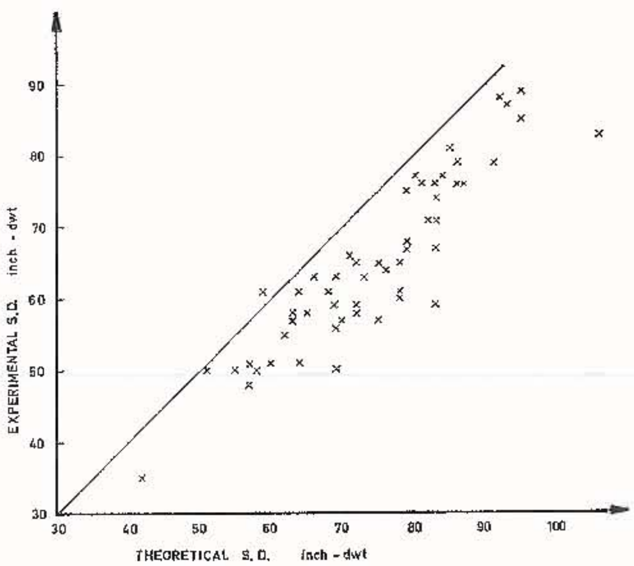

Fig. 2. A comparison of theoretical and experimental kriging standard deviations.

No sample in the estimated panel

The kriging estimator has a smaller s.d. than the inverse square weighting estimator in all the model orebodies produced. The difference between the two s.d.'s varies between one to 11 in-dwt. The mean deviations of both esitmators for an orebody are almost identical to each other and to those obtained in the previous case.

On average the experimental difference between the two central kriging estimators, with and without a central sample, is 2 in-dwt, and the corresponding difference between the theoretical s.d.'s is 5 in-dwt. This means that the theoretical s.d. is again an overestimate of the experimental s.d.

LARGE PANELS

3 × 3 panels

The s.d. of the kriging estimate is on average 3 in-dwt lower than that of both Sichel's estimator and the arithmetic mean of the nine samples. The mean deviation of the kriging estimate over the whole of a single orebody varies between −11 and +12 in-dwt, with an average over the 56 orebodies of zero.

A comparison between the theoretical and experimental s.d.'s of the kriging estimate shows that on average the theoretical s.d. underestimates the experimental results slightly.

Other large panels

As the size of a panel increases the number of samples in the panel increases, and so the s.d.'s of the estimators decrease. Thus, the difference between the estimators becomes less.

CONCLUSIONS

(i) It appears that in all cases the kriging estimator has the minimum variance.

(ii) It has been observed that there is a slight tendency for the kriging estimator to overvalue low-grade blocks and to undervalue high-grade ones. This is being investigated at present.

(iii) The theoretical kriging variance is a satisfactory estimate of the experimental variance.

(iv) The confidence limits for the kriging estimator are 0.36 times as wide as those for a single sample estimator. However, taking the average over 56 orebodies, the s.d. of a single sample estimator is 1.7 times greater than the s.d. of all the samples in an ore body. Thus, the s.d. of all the samples within an orebody does not provide a good estimate of the s.d. of a single sample estimator.

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