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Analysis of Cutoff Grades Using Optimum Control Theory

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The determination of the optimum cutoff grade for a surface mine may be very different from that of an underground mine because a selectable mining method may be used in the latter case. A review of research into the theory of cutoff grade is made and then a new method, based on Optimal Control Theory, is presented. This method uses a Hamiltonian function to assist in obtaining the solution. It is shown that the life of a mine can be divided into three parts: a beginning period of planning before mining; an intermediate period when both an average price and market information are used continually to adjust the cutoff grade; and, a final period when the main consideration is the current market situation.

Introduction

The selection of the optimum cutoff grade for an ore deposit is a fundamental problem facing the mining engineer. Important contributions to this topic have been made by Lane[1] and Henning.[2] Lane showed how to select the cutoff grade to maximize the total present value over the life of the mine. His method considers what may be called an opportunity cost which is affected by the discount rate. Later, Taylor[3] and Schaap[4] enlarged upon this concept. Elbrond,[5] Dowd,[6] Rudenno,[7] Jordi and Currin[8] and Napier[9] all considered the problem of maximizing the net present value (NPV).

To illustrate the traditional approach, consider any arbitrary statistical distribution, $f(x)$, of the ore grades in a deposit. This is represented in Figure 1.

The average grade $\bar{x}$ is

$$\bar{x} = \int_{-\infty}^{+\infty} x\, f(x)\, \mathrm{d}x. \qquad [1]$$

If a cutoff grade g is selected and no ore below this is mined (or, if mined, not sent to the mill), then the average grade will be increased. It is:

$$E(g) = \int_g^{\infty} x\, f(x)\, \mathrm{d}x \,/\, \int_g^{\infty} f(x)\, \mathrm{d}x \qquad [2]$$

The percentage of the original block mined is

$$T(g) = \int_g^{\infty} f(x)\, \mathrm{d}x. \qquad [3]$$

It is clear from Equations [2] and [3] that as the cutoff grade is increased two things happen: firstly, the average grade of the ore sent to the mill increases (Eq. [2]); secondly, the percent of the orebody mined decreases (Eq. [3]). Charts and tables for selected statistical distributions such as the normal or lognormal exist.[10] However, it is possible to program these equations for any distribution.

In many previous investigations no distinction has been made between the various mining methods. In general, a deposit will be mined either by open pit means or by an underground method. The optimum cutoff grade policy may vary with the mining method. This is because an open pit mine is worked in one basic way whereas an underground mine may be developed in several ways. For example, in an open pit operation as the market price rises, the cutoff grade should be reduced to increase NPV. However, in an underground mine, with selective mining methods, the cutoff grade has to be increased to maximize the profits if the price of the metal increases.

This paradox can be understood by considering that for an open pit operation all material is normally mined. As the cutoff grade is decreased, more ore may be sent to the mill. But in underground mining, it is possible to mine selectively rich ore to increase the output of the final refined metal. In this paper, only the case of an underground mine is considered.

Elbrond,[5] Rudenno[7] and Jordi and Currin[8] have used, respectively, dynamic programming, linear programming and non-linear programming to solve the problem of the optimum cutoff policy.

Nilsson and Aaro[11] have shown that the size of a deposit affects the optimum cutoff grade. However, the cost associated with the rate of development has not been considered to date. In addition, the fact that it may be possible to mine selectively the deposit and then return

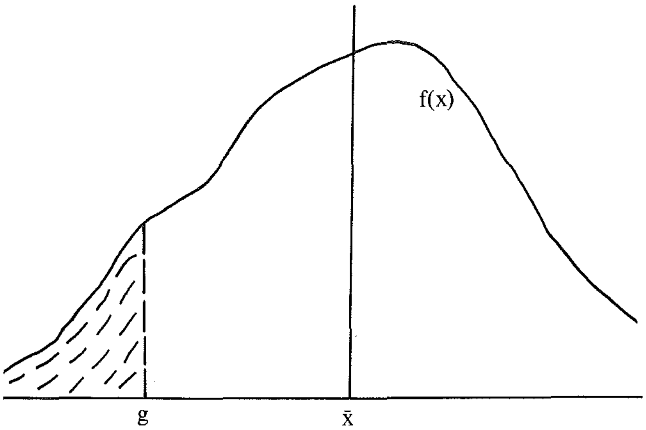

FIGURE 1. Arbitrary statistical distribution of ore grades

at some later date to complete the mining has been neglected. Both these factors will be incorporated into the model presented here.

Basic assumptions

1. The objective in the planning stage is to maximize the NPV by mining the whole orebody under an average price for the product.
2. The objective in the operating stage is to maximize the NPV of profit each year by mining the remaining orebody under the current price and some selected average future market price.
3. The orebody is homogeneous and the distribution of ore grade known.
4. There is no constraint on capacity at the refining stage.
5. The concentrator and smelter have a constant recovery.
6. The tax rate on profits will not change.
7. Salvage value of the equipment can be neglected.
8. Mining costs consist of two parts: one is a fixed cost per unit time and the other a variable cost per unit of ore.
9. The orebody can be mined selectively.

An important characteristic of mining underground with selective methods is that if ore is equal to or above an economical cutoff grade it will be mined out, or it can be left in place. Clearly, the input rate of the smelter must keep up with the output rate of the concentrator. The input to the system is the raw material, namely, the ore, while the output, the final refined metal, depends on the market price and is constrained further by the capacity of both the mining rate and the concentrator. The input of ore is controlled by the production rate and the cutoff grade. Because the market conditions will vary with time, the system is dynamic.

Mathematical model

In underground mining, development has to be carried out in advance of mining. The development rate $W(t)$ and the mining rate $U(t)$ are considered as being equal: Let X be the total quantity of reserve, which includes the ore mined and the material not mined, from $t = 0$ to some time f. The ore mined out and the cutoff grade is some function of time, $g(t)$. The NPV will then be composed of three basic terms:

NPV = −(initial costs) + (profits/time) − (mining costs/time).

These three terms are given by:

(a) the initial costs which are:

$$a + b\,U_o$$

where

a, b = are constants
U_o = the production rate at the time zero

(b) profits/time

$$-\ C_o + [(R(t) - Cs)\ U(t)\ E(g(t))]\ Mu$$

where

C_o = fixed costs per unit time.

(i.e. fixed costs for mining, developing, concentrating and refining)

$R(t)$ = market price/unit of refined metal
Cs = variable costs per unit of refined metal
$T(g)$, $E(g)$ = distribution function of tonnage of ore and average grade (see Eq. [3])
Mu = recovery factor (%)

(c) The mining costs/time unit

$$(Cm + Cd + Cc)\ U(t)$$

where

Cm, Cd, Cc = the costs of mining, developing and concentrating respectively.

In order to consider the NPV it is necessary to sum up and discount the last two terms:

$$\text{NPV} = -(a + b\,U_o) + \int_0^{tf} [-C_o + (R(t) - Cs)\ U(t)\ E(g(t))\ \text{Mu} - (Cm + Cd + Cc)\ U(t)]\ e^{-it}dt. \quad [4]$$

subject to:

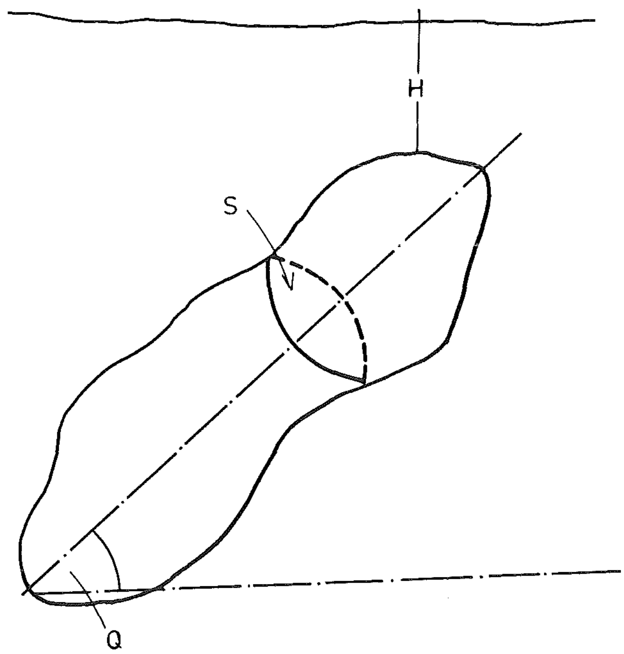

FIGURE 2. Geometry of inclined deposit

$$\frac{dx}{dt} = \frac{U(t)}{T[g(t)]}$$
$$X(0) = 0$$
$$X(tf) = Y$$
$$U_{min} \le U(t) \le U_o$$
$$g_{min} \le g(t) \le g_{max}$$

where

U_{min} = lowest limit for production

g_{min}, g_{max} = lowest and highest limits for cutoff grades.

$T(g)$ and $E(g)$ are as defined in Equations [1], [2] and [3]. In the event that the ore deposit is inclined or vertical, the fact that the mining costs increase with depth should be included in Equation [4]. Figure 2 illustrates this situation.

Equation [4] can then be written as:

$$\text{NPV} = -(a + b\,U_o) + \int_0^{tf} [-C + (R(t) - Cs)\,U(t)\,E(g)\,Mu - (Cm + Cd + Cc + af.\,H + B\,X)\,U(t)]\,e^{-it}dt \quad [5]$$

where

af = $af0/100$

B = $af\,|\sin Q\,|/S/r$

$af0$ = cost per 100 metres for developing

H = vertical distance from surface to top of deposit

S = cross-sectional area of deposit

r = specific gravity of ore

Basic solution for a horizontal deposit

Equation [4] represents a model of horizontal deposits. It can be solved by optimal control theory[12] by introducing a special function called the Hamiltonian. This function will have the form:

$$H = -C_o + (R - Cs)\,U\,E(g)\,Mu - (Cm + Cd + Cc)\,U + Nmt\cdot U/T(g) \quad [6]$$

when Nmt is an adjoint variable. It is a function of time.

This Hamiltonian is a linear function of $U(t)$. So $U(t)$ has the form:

$$U = U_o \qquad F(t) > 0$$
$$U = U_{min} \qquad F(t) < 0$$
$$U = \text{undetermin} \qquad F(t) = 0$$

where

$$F(t) = [R(t) - Cs]\,E(g)\,Mu - (Cm + Cd + Cs) + Nmt(t)/T\,[g(t)]. \quad [7]$$

Since for maximum, $\partial H/\partial g = 0$, we have

$$Mu(R - Cs)\,T(g)dE(g)/dg + Nmt\cdot T(g)d[1/T(g)]/dg = 0. \quad [8]$$

Equations [2] and [3] can be combined to yield:

$$T(g).E(g) = \int_g^1 ak.f(ak).dak \quad [9]$$

and taking derivatives gives

$$T(g).dE/dg = [E(g) - g]f(g)$$

Also, from [1]

$$dT(g)/dg = -f(g)$$

Thus, Equation [8] becomes

$$(R - Cs)[E(g) - g].T(g).Mu = -Nmt \quad [10]$$

Substituting [10] into [7] yields

$$F(t) = [R(t) - Cs].g(t)Mu - (Cm + Cd - Cc) \quad [11]$$

with $F(t) = 0$ it follows that

$$g(t) > (Cm + Cd + Cc)/[R(t) - Cs] \quad F > 0$$
$$g(t) < (Cm + Cd + Cc)/[R(t) - Cs] \quad F < 0$$
$$g(t) = (Cm + Cd + Cc)/[R(t) - Cs] \quad F = 0 \quad [12]$$

Equation [10] yields an expression for $Nmt(t)$:

$$[E(g) - g].T(g) = -Nmt(t)/[R(t) - Cs]Mu. \quad [13]$$

A boundary condition for the problem is that

$$Nmt(tf) = K$$
$$dNmt/dt = i.Nmt - H/X = i.Nmt \quad [14]$$

The solution to [14] is

$$Nmt = Ke^{i(t - tf)}$$

Substitution into Equation [13] yields:

$$[E(g) - g].T(g) = \frac{-Ke^{i(t - tf)}}{(R - Cs)Mu} \quad [15]$$

At time $t = tf$ [13] becomes

$$K = (Rtf - Cs)\,Mu\,[E(gtf) - gtf]T(gtf) \quad [16]$$

where

gtf = the cutoff grade in year tf

Rtf = the market price in time tf.

Substitution of [16] into [15] gives

$$[E(g) - g].T(g) = \frac{Rtf - Cs}{R(t) - Cs}\,[E(gtf) - gtf]T(gtf)e^{i(t - tf)} \quad [17]$$

Returning to [6] and using the fact that if tf is optimum, then $H(tf) = 0$ yields:

$$-C_o + (Rtf - Cs)\,E(gtf)Mu.Utf - (Cm + Cd + Cc)Utf + Nmt.tf.Utf/T(gtf) = 0 \quad [18]$$

since $Nmt.tf = K$

$$gtf = (C_o/Utf + Cm + Cd + Cc)/[Rtf - Cs)Mu] \quad [19]$$

if $gtf < g_{min}$, then $gtf = g_{min}$;
$gtf > g_{max}$, then $gtf = g_{max}$.

We can now draw conclusions from the model.

Case of constant market price

If the market price of the mineral is constant, then

$$R(t) = Ra = Rtf$$

where

Ra = the constant price.

Equation [17] is

$$[E(g) - g]T(g) = (E(gtf) - gtf)T(gtf)e^{i(t - tf)}$$
$$g(t) > g_{max},\ g(t) = g_{max}$$
$$g(t) < g_{min},\ g(t) = g_{min} \quad [20]$$

Equation [11] is

$$F(t) = (Ra - Cs)\, g(t)Mu - (Cm + Cd + Cc) \qquad [21]$$

Suppose $gtf < g_{max}$ and since

$$gtf = \max\{(C_o/Utf + Cm + Cd + Cc)/[(Ra - Cs)]Mu,\ g_{min}\} > (Cm + Cd + Cc)/[(Ra - Cs)Mu]$$

with

$$F(tf) > 0,\ Utf = U_o$$

$$gtf = (C_o/U_o + Cm + Cd + Cc)/[(Ra - Cs)Mu]. \qquad [22]$$

Since $0 < g < 1$, $T(g)\,[E(g) - g]$ is a monotonically decreasing function of g. From [20] with gtf given, g should decrease with time, and there follows: $F > 0$. This means that if the mining life is optimum and the price of the metal a constant, the production rate must be constant and take on a value as large as possible. Applying a discount rate makes the cutoff grade decrease with time in such a way that it reaches its lowest value at the end of the mining life. This conclusion has been reached by Henning[2] but his assumptions should be modified appropriately.

Equation [20] also leads to other significant economic conclusions:

1. $gtf < 0$
   For this to happen, $Ra < Cs$. Thus the price cannot even cover the cost of refining. No ore will be mined.
2. $gtf \geq 1$
   The average revenue is less than or equal to all of the fixed and variable costs. Mining the ore will give a loss or no profit.
3. $0 < gtf < g_{min}$
   The orebody has a very high economic value but some ore must remain unmined.
4. $g_{max} < gtf < 1$
   The grade of the ore is too low to mine.

Equations [20], [21] and [22] give not only an optimum control policy for the cutoff grade but a necessary condition for mining the ore.

Variable market price

This is the most interesting case where the market price of the metal will change with time. The relationship between the cutoff grades for time increasing from t to $t + \Delta t$ is found from Equation [17] with (at first) $i = 0$:

$$\{E[g(t + \Delta t)] - g(t + \Delta t)\}\, T[g(t + \Delta t)] = \frac{R(t) - Cs}{R(t + \Delta t) - Cs}\cdot\{E[g(t)] - g(t)\}\, T[g(t)] \qquad [23]$$

From [23] it follows that if the market price in time $t + \Delta t$ is higher than in time at t, then

$$g(t + \Delta t) > g(t)$$

Therefore, as the market price rises, the cutoff grade should increase. The physical meaning of this is that when the price increases, richer ore should be mined to increase the total profit as quickly as possible. As the price is lower, more poor ore is to be mined which, in effect, saves the higher grades for better economic times.

If the discount rate, i, is taken into account, this conclusion is modified. In this case the cutoff grade will increase with rising price only if the increased value of the metal can compensate for the decrease value caused by the discount rate.

General solution to the model

In order to obtain a solution to the model, the mine will be considered as being in two stages: a planning stage when the market price is assumed constant; an operating stage where the market price will fluctuate with time. The following equations must be solved for the planning stage:

$$[E(g) - g]T(g) = [E(gtf) - gtf]T(gtf)e^{i(t - tf)}$$

$$gtf = (C_o/U_o + Cm + Cd + Cc)/[(Ra - Cs)Mu]$$

$$U(t) = U_o$$

$$\sum_{t=0}^{tf} U_o/T[g(t)] = Y$$

$$g(t) < g_{min} \quad g(t) = g_{min}$$

$$g(t) > g_{max} \quad g(t) = g_{max}.$$

The following steps are followed:

1. Calculate gtf
   if $gtf < g_{min}$, $gtf = g_{min}$
   if $gtf > g_{max}$, $gtf = g_{max}$
2. Let $tf - t = 0, 1, 2, \ldots m, \ldots$ and determine $g_i (i = 0, 1, 2, \ldots m, \ldots)$
3. If there is a $g_m > g_{max}$, then
   $g_m = g_j = g_{max} \quad j = m + 1, m + 2, \ldots$
4. Calculate Z_n

$$Z_n = U_o/T(gtf) + \sum_{i=1}^{m} U_o/T(g_i) + \sum_{j=m+1}^{n} U_o/T(g_{max})$$

5. If in step n there is $Z_n > Y$ stop. Then $tf = n$.

Thus at the end of the calculation the optimum life is determined. If U_o is not given, it is necessary to assume different values for it and repeat the steps serially until an optimum value for U_o is obtained.

In the operating stage the market price fluctuates. However, as time continues information can be obtained to make predictions regarding the price of the metal. To handle this situation, two time periods are considered:

1. in the first period an average price is used to determine an optimum mining life (as was done in the planning stage);
2. using the current price and the past average price, a new cutoff grade and production are determined from the following system of equations:

$$[E(g) - g]T(g) = \frac{Ra - Cs}{R(t) - Cs}$$

$$[E(gtf) - gtf]T(gtf)e^{i(t - tf)}$$

$$gtf = [(C_o/U_o + Cm + Cd + Cc)/[(Ra - Cs)Mu],\ g_{min},\ g_{max}]$$

$$g(t) < g_{min},\ g(t) = g_{min}$$

$g(t) > g_{max}$, $g(t) = g_{max}$
$U(t) = U_o F(t) > 0$
$U(t) = U_{min} F(t) < 0$
$U(t) =$ undetermin $F(t) = 0$
$F = [R(t) - Cs].g(t).Mu - (Cm + Cd + Cc)$.

The first period usually lasts for three to five years. In the equation above it means that revenues just cover mining costs. If the future outlook is good, production levels today are kept small.

For the second period, the following steps are to be followed:

1. Use the information obtained in the first period to calculate a new future average price, $Ra1$.
2. Calculate the remaining ore to be mined.
3. Use $Ra1$ and $Y1$ to determine a new mining life $tf1$ by the same method used the planning stage.
4. Use the present market price $R(t)$, the new average price $Ra1$, the mine life $tf1$, and the deposit size $Y1$, to determine the cutoff grade and production by the method given for the first period.
5. Use the results of 4 to calculate yet another average price, $Ra2$, deposit size, $Y2$ and mining life, $tf2$. Repeat steps 2 to 5 until the deposit is exhausted.

Then, in a step-by-step process, as more information is available, it is possible to make the mining system as near to the optimum state as possible.

Worked example

A computer program was written to model the equations presented here. The program can be used for both the planning and operating stages. The following are sample data:

Frequency distribution

cutoff	0.00	0.02	0.04	0.06	0.08
freq.	0.05	0.15	0.50	0.20	0.10

Prices and costs
Ra = \$1000/ton of material refined
C_o = \$200 000/year
Cs = \$10/ton of mineral refined
Cc = \$8/ton of ore
Cm = \$20/ton of ore
Cd = \$2/ton

Constants
a = \$3 000 000
b = \$30/ton of ore

Other factors
Y = 10 000 000 tons
i = 15%
Mu = 0.95

Constraints for the cutoff and production rate
$g_{max} = 0.08$
$g_{min} = 0.01$
U_{max} = 500 000 ton/yr
U_{min} = 100 000 ton/yr

The results are presented in the Appendix. Table 1 is for the planning stage, and Tables 2 and 3 are for the operating stage.

Vertical deposit

Equation [5] represents the case of a vertical model. The same approach as used previously can be followed here. The Hamiltonian now has the form:

$$H = -C_o + (R - Cs).Mu.E(g)\,U - (Cm + Cd + Cc + af.H + B.X)U + Nmt.U/T(g).$$

$U(t)$ takes the form:

$U = U_o \quad F(t) > 0$
$U = U_{min} \quad F(t) < 0$
$U =$ undetermin $\quad F(t) = 0$

where

$$F(t) = [R(t) - Cs]E(g(t))Mu + (Cm + Cd + Cc + af.H + B.X(t) + Nmt(t)/T(g(t)).$$

In the planning stage the market price is taken as a constant. If the mining life and the size of deposit are both optimum it can be shown that there must be

$$U(t) = Utf = U_o.$$

Thus the equation analogous to [17] becomes:

$$[E(g) - g]T(g) = \{[(Ra - Cs)Mu[E(gtf) - gtf]T(gtf) - BU_o/i]e^{i(t - tf)} + BU/i\}/\{[R(t) - Cs]Mu\}$$
$$gtf = \{(C_o/U_o + Cm + Cd + Cc + af.H + B.Y)/[(Ra - Cs)Mu], g_{min}, g_{max}\}$$
$g(t) < g_{min} \quad g(t) = g_{min}$
$g(t) > g_{max} \quad g(t) = g_{max}$
$U = U_o \quad F(t) > 0$
$U = U_{min} \quad F(t) < 0$
$U =$ undetermin $\quad F(t) = 0$
$$F(t) = [R(t) - Cs]g(t)Mu - [Cm + Cd + Cc + af.H + B.X(t)]$$
$$\sum^{t} U/T(g) = X$$
$$\sum^{t} U/T(g) = Y. \quad [24]$$

The difficulty in determining $g(t)$ at the operating stage is that U is contained in the expression above. However, an iterative approach can be followed. First, let $U = U_o$ and determine g_1. Put this value of g_1 into $F(t)$ to see if $F(t) > 0$. If so, then g_1 and U_o are the optimum solution. If not, set $U = U_{min}$ and determine g_2. Put this value of g_2 in $F(t)$. If $F(t) < 0$, the solution is $g = g_2$, $U = U_{min}$. If this is not the case, the solution cannot be determined and a decision must be made. If the future price of the metal may be higher, use a low production level, otherwise a higher level. After this production level is (somehow) estimated, it is used to determine g.

Influence of mining cost in the cutoff control policy

The mining cost af for development can have a significant influence on the cutoff sequence. From Equation [24] let $Ra = R(t)$ and $U(t) = U_o$, and we can get

$$V = [E(g) - g]T(g) = [E(gtf) - gtf]T(gtf)e^{i(t - tf)} + B.U_o/[i(1 - e^{i(t - tf)}]/[(Ra - Cs)Mu].$$

The first term on the right will make V increase as time t increases. This results in the cutoff grade decreasing with time. The second term makes the value of V decrease with time hence increasing the cutoff grade. It is possible that as B reaches a certain point the increase due to the first term and the decrease due to the second will cancel each other. If B exceeds this point, the cutoff may increase with time. This will often be the case for inclined or vertical orebodies since mining costs will increase with depth. The way to compensate for these increased costs is to mine higher grade ore.

Another example

The model based on Equation [24] is used in a computer program to obtain the results presented in the Appendix. The following input data are added to the previous example.

$af\,0$ = \$6/ton/100 metres
S = 6000 m^2
H = 40 m
R = 3 t/m^3
Q = 90°

The parameters $af\,0$ (Table 4) and Q (Table 5) are changed to illustrate the effect on the cutoff policy in the planning stage.

Leaving ore to be mined later

The model used here allows for the selective mining of different areas. If ore is left in an area, it must contain sufficient values to justify its being mined out in the future. If the cost of mining the remaining ore increases, the ore left behind will not be mined, as the previous section showed how the cutoff grade will combine to increase. The reason the ore was left behind in the first place was because its grade was too low.

Suppose the cost of mining becomes low or the orebody is nearly horizontal. Then the cutoff will decrease. Also, suppose the deadline to begin mining an area left behind is two years. Then a mining policy can be determined by the following steps:

1. Determine a time T for the start of mining this area (here, T is arbitrarily taken as 2 years).
2. Use the same method as in the first section (planning stage) determine a mining life $tf0$ and the cutoff and production rate for each year as $g0$ and $U0$.
3. Use the following equations to determine the value of the state variables for each year.

   In year 1,

   $$H_1 = U_1 \quad X_1 = H_1/T(g_1).$$

   In year 2,

   $$H_2 = U_2 - H_1/T(g_1) \int_{g_2}^{g_1} f(g)\, dg$$
   $$X_2 = H_1/T(g_1) + H_2/T(g_2).$$

   In year 3,

   $$H_3 = U_3 - \{H_1/T(g_1) \int_{g_3}^{g_2} f(g) dg + H_2/T(g_2) \int_{g_3}^{g_2} f(g)\, dg\}$$
   $$X_3 = H_1/T(g_1) + H_2/T(g_2) + H_3/T(g_3),$$

   and if the deadline is To years the state in year n can be determined by:

   $$H_n \doteq U_n - \{H_{n-To}/T(g_{n-To}) \int_{g_n}^{g_{n-1}} f(g)\, dg + \ldots + H_{n-1}/T(g_{n-1}) \int_{g_n}^{g_{n-1}} f(g)\, dg$$
   $$X_n = H_1/T(g_1) + H_2/T(g_2) + \ldots + H_n/T(g_n)$$
   $$H_1 = U_1.$$
4. If $X(tf0) < Y$ then determine a new mining life $tf1$ by increasing $tf0$ by 1 and use this new life to obtain a new group of cutoff $g1$ and production rate $U1$.
5. Repeat steps 3 to 4 until $X(tfn)$ is equal to or larger than Y.

Conclusions

(a) Cutoff should be increased as the price rises. But the cutoff should only be increased as the increased price covers the decreased value caused by the discount rate.
(b) Discount rate makes the cutoff decrease with time but the cost due to increased mining depth makes the cutoff increase with time.
(c) If the market price is static and the mining life is optimum, production rate must be constant and as high as possible.
(d) If the market price is fluctuating the determination of production rate should follow the rule: as the marginal revenue per unit of marginal ore is larger than the marginal cost, there must be a highest possible production level, otherwise the lowest level.

References

1. LANE, K.F. Choosing the optimum cutoff grade. *Col. Sch. of Mines Quar.*, vol. 59, 1964. pp. 811–829.
2. HENNING, U. Calculation of cutoff grade. *Can. Min. J.*, vol. 84, March 1963. pp. 54–57.
3. TAYLOR, H.K. General background theory of cutoff grades. *Trans. Inst. Min. Met.*, Section A, vol. 81, 1972. pp. A160–179.
4. SCHAAP, W. Objective function and methodology in cutoff grade theory. *Proceedings, 18th APCOM*, London, 1985. pp. 493–502.
5. ELBROND, J., DUBOIS, J.F. and DAOUST, C.T. Rate of production and cutoff grade: a program system for teaching and experimentation. *15th APCOM Symposium*, Aus. I.M.M., Brisbane, Australia, 1977. pp. 13–19.
6. DOWD, P.A. Application of dynamic and stochastic programming to optimize cutoff grades and production rates. *Trans. Inst. Min. Met.*, Section A, vol. 85, 1976. pp. A22–31.
7. RUDENNO, V. Determination of optimum cutoff grades. *Proceedings, 16th APCOM, 1973*, Tuscon, Arizona. Am. Inst. of Min. Engrs. pp. 261–268.
8. JORDI, K.C. and CURRIN, D.C. Goal programming for strategic planning. *Proceedings, 16th APCOM*, 1979, Tucson, Arizona. Am. Inst. of Min. Engrs. pp. 296–303.
9. NAPIER, J. The effect of cost and price fluctuations on the optimum choice of mine cutoff grades. *J. S.*

Afr. Inst. Min. Met., June 1983. pp. 117 – 125.
10. DAVID, N. *Geostatistical Ore Reserve Estimation.* Elsevier, Amsterdam, 1977.
11. NILSSON, D. and BENGT AARO. Cutoff grade optimization – a question of bottlenecks and ore reserves. *Inter. Min.*, July 1985. pp. 28 – 33.
12. SURESH, P.S. and THOMPSON, G.L. *Optimum Control Theory – Applications to Management Science.* 1981.

Appendix

Example 1

TABLE 1.

t	g	ga	t	g	ga
1	.0580	.0650	7	.0453	.0544
2	.0566	.0638	8	.0419	.0516
3	.0550	.0625	9	.0392	.0498
4	.0531	.0609	10	.0375	.0494
5	.0509	.0590	11	.0354	.0489
6	.0483	.0569	12		
tf*	10.7	UO	500000	NPV	44306868
Y*		10000000		Ra	1000

TABLE 2.

t	R	g	ga	U	APV
1	1000	.0576	.0647	500000	12055516
2	1050	.0566	.0639	500000	11220659
3	1300	.0573	.0644	500000	14699933
4	650	.0447	.0539	100000	–90153
Ra		1000		Yout	4193143

TABLE 3.

t	R	g	ga	U	APV
5	1000	.0519	.0599	500000	5560934
6	700	.0451	.0543	100000	47712
7	500	.0100	.0441	100000	–484900
8	600	.0100	.0441	100000	–291082
9	900	.0533	.0611	500000	2442108
10	1200	.0538	.0615	500000	4098772
11	1200	.0497	.0581	500000	3152240
12	900	.0399	.0500	500000	781258
13	800	.0374	.0494	100000	37900
14	900	.0386	.0497	500000	562903
Ra		907.143		NPV	35793802
Yout		9969101		tf*	14

Note:
t = time
g = cutoff grade
ga = average grade
tf* = optimum mining life
Y* = optimum reserve
UO = mining capacity
R = market price of the mineral
Ra = average price
NPV = total net present value
PV = present value

Example 2

TABLE 4.

af0	6.00	3.00	2.00	1.00
1	.0502	.0497	.0519	.0556
2	.0505	.0498	.0516	.0549
3	.0508	.0499	.0513	.0541
4	.0511	.0501	.0509	.0531
5	.0515	.0503	.0505	.0520
6	.0520	.0505	.0500	.0507
7	.0526	.0507	.0494	.0491
8	.0532	.0510	.0488	.0474
9	.0540	.0514	.0480	.0453
10	.0549	.0517	.0471	.0429
11	.0559	.0522	.0460	.0401
12	.0571	.0527		.0388
13	.0585			
14	.0602			
15	.0658			
tf*	15.0	11.7	10.9	11.0
Y*	8000000	10000000	10000000	10000000
UO	230000	450000	500000	500000
NPV	11904340	27548624	35528876	43058776

TABLE 5.

Q	90	30	20	10
1	.0502	.0504	.0521	.0550
2	.0505	.0505	.0519	.0542
3	.0508	.0507	.0517	.0533
4	.0511	.0510	.0515	.0523
5	.0515	.0512	.0512	.0511
6	.0520	.0515	.0509	.0497
7	.0526	.0519	.0505	.0480
8	.0532	.0523	.0500	.0461
9	.0540	.0527	.0495	.0439
10	.0549	.0533	.0489	.0413
11	.0559	.0539	.0483	
12	.0571			
13	.0585			
14	.0602			
15	.0658			
tf*	15.0	11.5	10.5	10.3
Y*	8000000	10000000	10000000	10000000
UO	230000	440000	500000	500000
NPV	11904340	25082656	31485732	38253840

U = production rate
Yout = ore reserve mined
af0 = cost increased by the mining depth
Q = dip of the orebody