Tenth international symposium on the application of computer methods in the mineral industry (APCOM)

Many of the papers to be presented at the Symposium will be of general interest to members. Arrangements have therefore been made to publish a number of these papers in the Journal. The first of these are the papers on 'Simulation of Crushing Plants' and 'Computer Applications in Rock Mechanics' published this month.

The Simulation of Crushing Plants with Models Developed using Multiple Spline Regression

By W J Whiten,* B.Sc.

SYNOPSIS

Models of cone crushers and vibrating screens are described. These models have been constructed from actual plant data using a systematic model-building technique. Included in this technique is regression analysis using multi-dimensional spline functions.

The models have been used too simulate complete crushing plants and hence to design and evaluate a control system for crushing plants.

SINOPSIS

Modelle van koniese vergruisers en vibreersiwwe word beskrywe. Hierdie modelle was gekonstrueer van informasie verkry van werklike aanlegte deur die gebruikmaking van 'n sistematiese model-bou tegniek. Die modelle was gebruik om volledige vergruisings aanlegte na te boots en sodoende 'n kontrole sisteem vir vergruisings aanlegte te ontwerp en die waarde daarvan te bepaal.

INTRODUCTION

A research group now known as the Julius Kruttschnitt Mineral Research Centre at the University of Queensland has been working since 1962 on the simulation, optimization and control of mineral treatment processes. The initial work was on the grinding and classification processes. This work has been very successful in the optimization (Lynch, *et al*, 1967) and control (Draper, *et al*, 1969) of grinding circuits. This paper shows how the techniques which were applied to grinding have been extended and applied to crushing plants. Models of cone crushers and vibrating screens have been developed from data taken from an operating industrial plant. These models have been used for simulation of complete crushing plants and have been used to aid the design of a control system for a crushing plant and to study the behaviour of the plant under this control system.

---

*Research Officer, Julius Kruttschnitt Mineral Research Centre, University of Queensland, Brisbane, Australia.

DATA COLLECTION

Test work was conducted on the fine crushing plant of the No. 2 Concentrator at Mount Isa Mines, Limited, during July, 1968. This plant contained the following equipment:

Primary vibrating screen
: 6-ft × 12-ft single-deck Allis-Chalmers with rod deck.

Primary crusher
: 7-ft Symons short-head crusher with extra-coarse cavity bowl.

Secondary vibrating screens (four)
: 6-ft × 16-ft single-deck Allis-Chalmers with square mesh cloths.

Secondary crushers (two)
: 7-ft Symons short-head crusher with fine cavity bowl.

Three surge bins of approximately 100 tons capacity are also included in the plant. A flow diagram of the plant is given in Fig. 1.

As the aim of these tests was to collect data for the development of steady-state models, it was necessary

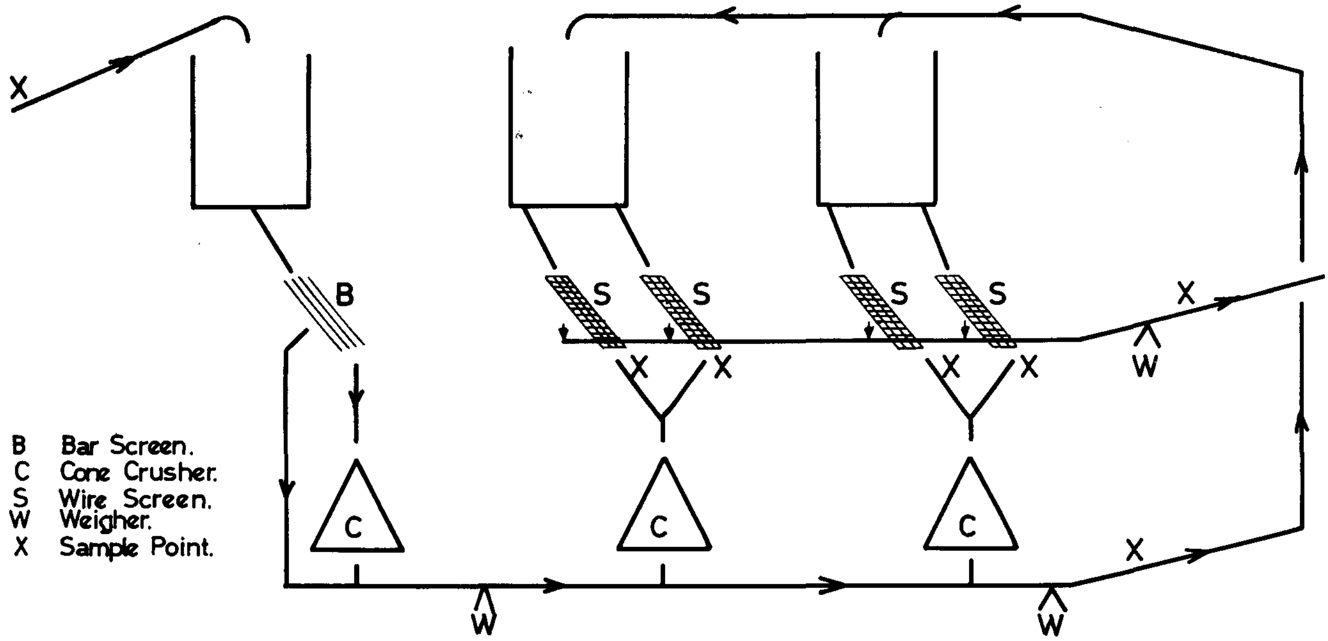

Fig. 1—Flow diagram of fine crushing plant

that the plant be operating steadily during the collection of data. The presence of two 100-ton surge bins in the closed circuit meant that the plant could run for an hour or more under conditions significantly different from steady state. This, together with frequent minor stoppages, meant that the plant seldom ran under steady-state conditions. In addition to these variations, the discharge rate from the crushers contained considerable short-term variations.

At the start of the data collection, tests of the behaviour of the entire plant were attempted. The later test work concentrated on the individual units as a wider range of operating conditions could be obtained for these by using the surge bins to provide feed during the sampling. The data collected consisted of size distributions, ore tonnages, crusher current, crusher gap and screen sizes. Figure 1 shows the sample points. These data were obtained in tests on the individual units for the purpose of model building.

The primary section

It was not possible to obtain samples to separate the behaviour of the primary screen and crusher but the data provide five tests of the behaviour of this complete section at tonnages ranging between 380 and 820 tons per hour. Due to experimental difficulties, an average feed sizing was assumed for all these tests.

The secondary screens

Ten tests were conducted on the screen behaviour. The screen sizes used were 0,5, 0,625 and 0,75 inches. The feed tonnages ranged from 219 to 420 tons per hour.

The secondary crushers

Thirteen tests of secondary crusher behaviour were obtained. The crusher gap ranged from 0,2 to 0,5 in. and the feed tonnages from 120 to 360 tons per hour.

THE CONE CRUSHER MODEL

A simple model has been used for the cone crusher as it was felt that more complex models, which could easily be proposed, would contain more parameters than could be calculated using the small amount of available data which were, of course, of limited accuracy.

The basis of the cone crusher model is the assumption that particles can either be broken or dropped through the crusher unbroken. The broken particles then have the same choice of dropping through the crusher or of being broken further. Thus, the cone crusher is simplified to a single breakage zone and a probability of entering or re-entering this breakage zone. Figure 2 shows symbolically the parts of this cone crusher model and the internal flows between them. The vectors **f**, **x** and **p** give the flow rates in each size fraction. The lower triangular matrix **B** gives the relative distribution of each size fraction after it is broken and the diagonal matrix **C** gives the proportion of particles entering the breakage region.

The mass balances at the nodes in Fig. 2 give the equations

$$\mathbf{f}+\mathbf{BCx}=\mathbf{x} \quad \ldots \quad (1)$$

and

$$\mathbf{x}=\mathbf{Cx}+\mathbf{p} \quad \ldots \quad (2)$$

Eliminating **x** gives the equation

$$\mathbf{p}=[\mathbf{1}-\mathbf{C}]\ [\mathbf{1}-\mathbf{BC}]^{-1}\mathbf{f},$$

which expresses the crusher product in terms of the feed. The matrix $\mathbf{1}-\mathbf{BC}$ is always non-singular as a unit element on the diagonal of **BC** implies both no breakage and no discharge of that size fraction.

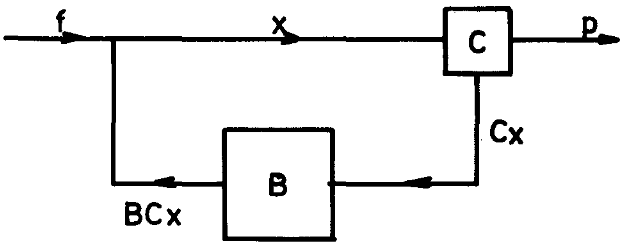

Fig. 2—Symbolic representation of crusher model

The breakage matrix

The observed data on the cone crusher appear to be explained best by a breakage matrix consisting of two parts. The first part is a step matrix $\mathbf{B}_1$ which gives the product sizes relative to the size of the original particle. This is calculated from

$$p(s)=\left(1-e^{-(s/s')^u}\right)\Big/\left(1-e^{-1}\right),$$

where s' is the size of the original particle and $p(s)$ is the fraction less than size s. This distribution is a modification of the Rosin-Rammler distribution given by Broadbent, *et al*, (1956). The value of u has been calculated to be $6{,}0\pm0{,}9$ for the cone crusher.

The second part of the breakage matrix, $\mathbf{B}_2$, describes the production of fines. The sizing of this portion of the product is not dependent on the size of the original particle. A Rosin-Rammler distribution is used to describe this material, that is,

$$p(x)=1-e^{-(s/s'')^v},$$

where s'' is a fixed size. In the calculations, this distribution is altered so that the material predicted above the size of the original particle appears after breakage at the original particle size. This amount is generally very small. The values calculated for the parameters in this distribution are

$s''=0{,}12\pm0{,}025$ inches,

$v=1{,}25\pm0{,}145$.

Finally, to get the total breakage matrix, $\mathbf{B}$, the two component matrices are added,

$\mathbf{B}=a\mathbf{B}_1+(1-a)\mathbf{B}_2$,

where a is calculated from the gap size in inches, g, as follows:

$a=0{,}872+0{,}115g\pm0{,}014$.

The standard deviations of the coefficients in this equation are 0,15 and 0,039. This equation shows, as would be expected, that more fine material is produced as the crusher gap is decreased.

The calculated breakage function indicates that when a particle is caught by the closing crusher gap it tends to produce a small number of large particles plus a few per cent of fines.

The classification matrix

The diagonal elements of the matrix $\mathbf{C}$ are obtained from a function of particle size, $c(s)$, which gives the probability of a particle of size s entering the breakage stage of the crusher model. It is assumed that particles below a certain size k_1 are not broken in the crusher, that is,

$c(s)=0{,}0 \quad$ for $s<k_1$.

Also, there exists a size k_2 above which particles are always broken, that is,

$c(s)=1{,}0 \quad$ for $s>k_2$.

Between these sizes $c(s)$ is assumed to be a parabola with zero gradient at k_2, thus,

$$c(s)=1-\left(\frac{s-k_2}{k_1-k_2}\right)^2 \qquad k_1<s<k_2.$$

The elements in the $\mathbf{C}$ matrix are obtained as the mean values of $c(s)$ in the appropriate size range. For the range s_1 to s_2 the matrix element is:

$$\int_{s_1}^{s_2} c(s)ds/(s_2-s_1).$$

The two parameters k_1 and k_2 are predicted by the equations

$k_1=0{,}67g\pm0{,}77$,

$k_2=1{,}121g+2{,}31q+T(t)\pm0{,}071$,

where g is the crusher gap in inches, q is the fraction plus one inch in the crusher feed and t is the feed tonnage to the crusher. The function $T(t)$ is a natural spline function of degree 3 (Ahlberg, *et al*, 1967) through the points

(100,0, $-0{,}0486$), (250,0, $-0{,}085$), (400,0, $-0{,}259$),

which is drawn in Fig. 3.

This relation for k_1 is as would be expected. The relation between k_2 and gap size is reasonable and the remaining terms in this equation appear to relate to the ease of flow through the crusher, that is, large particles and low tonnages flow through the crusher more rapidly.

The crusher current

This is calculated from the vector $\mathbf{Cx}$ which contains no fine particles and is related to the amount of breakage by

$\mathbf{f}-\mathbf{p}=[\mathbf{1}-\mathbf{B}]\mathbf{Cx}$,

obtained by adding equations (1) and (2).

A number a is defined by

$a=\Sigma t_i/(s_i+s_{i+1})$

where t_i is the i-th element of $\mathbf{Cx}$ and s_i, s_{i+1} are the upper and lower limits of the i-th size fraction. No problems arise in defining an average size of the fine

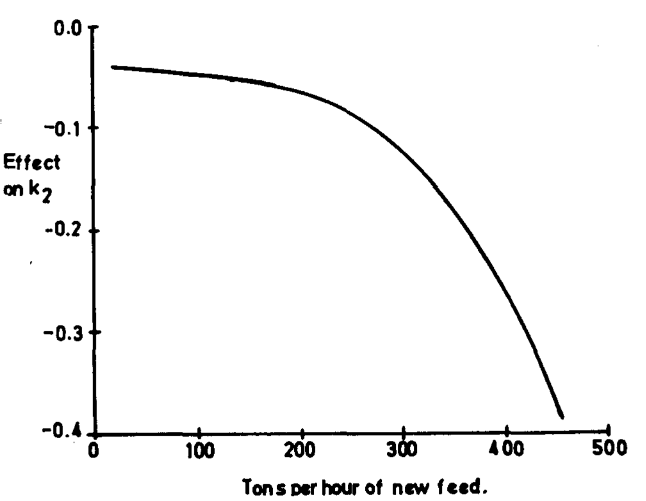

Fig. 3—Effect of feed rate on k_2 in the crusher model

size fractions as $t_i=0$ there. The crusher current is predicted by

$A=14{,}2+0{,}0822a+0{,}000\,305a^2\pm 1{,}8$

The complete cone crusher algorithm is given in Appendix A.

The accuracy of the model

The predictions of the cone crusher model have been compared with the data for each of the available tests. A mean error of 0.5 and a standard deviation of about 3,0 were obtained for the difference between the predicted and observed percentage sizings. It has been found that the error distribution can be reproduced quite accurately by adding random perturbations to the model parameters. These are

for k_1 $0{,}077\xi$
for k_2 $0{,}071\xi$
for σ $0{,}014\xi$

The ξ are independent random normal variates with zero error and unit standard deviations. Several simulations (say 20) need to be run using this technique to obtain estimates of the distribution of the differences between the model and the data.

THE VIBRATING SCREEN MODEL

This model has been developed from simple probabilistic considerations and the parameters found by fitting the model to the experimental data. The probability that a particle of size s does not pass through a screen with hole size h and wire diameter d is

$$E(s)=\left[1-\left(\frac{h-s}{h+d}\right)^2\right]^m$$

and this expression provides the efficiency curve for the screen. The number of trials, m, that a particle performs in crossing the screen is considered to be proportional to:

an efficiency constant $(k_1)^2$,
the length of the screen l, and
a load factor f.

Hence

$$m=k_1^2 l f.$$

The load factor f will be unity for low feed rates and tend to zero for very large feed rates. However, under the test conditions no effect of load was detected and hence, until data from heavily loaded screens are analyzed, f has been set at unity.

To use this model for the prediction of the amount of ore in size intervals, an average value of the efficiency curve, which becomes very steep, is required for each size fraction. The unweighted average value is

$$\int_{s_1}^{s_2} E(s)ds/(s_2-s_1) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

and the approximation

$$E(s) = e^{-m\left(\frac{h-s}{h+d}\right)^2}$$

may be used. Putting this approximation into equation (3) and making the substitution

$$y=\sqrt{m}\left(\frac{s-h}{h+d}\right)$$

gives

$$\frac{h+d}{\sqrt{m}}\int_{y_1}^{y_2} e^{-y^2}dy/(s_2-s_1).$$

This integral may be evaluated using the approximation

$$\int_y^{\infty} e^{-y^2}dy=0{,}124\,734/(y^3-0{,}437\,880\,5y^2+0{,}266\,982y+0{,}138\,375)$$

which is ERFC in Hart, *et al* (1968), and

$$\int_o^{\infty} e^{-y^2}dy=0{,}89$$

which are accurate to 0,01.

This model provides an adequate description of the screen behaviour, except for the sub-mesh material (minus 0,016 5 in.) of which the fraction k_3 is predicted to go to the oversize. The values of the model parameters are

$k_1=2{,}6\pm 0{,}4$
$k_3=0{,}10\pm 0{,}015$

No significant regression terms were found to predict variations in these values. However k_3 is known to become much higher ($\approx 0{,}3$) for a wet ore. Appendix B gives the complete vibrating screen simulation algorithm.

The accuracy of the screen model

The screen model has been used to simulate the experimental data. The percentage sizing data were reproduced with a mean error of 1,0 and a standard deviation of 2,0. The mean error in the predicted tonnages is 0,3 tons/h and the standard deviation 7,5 tons/h. As in the case of the cone crusher, the error distribution of the data can be reproduced accurately by perturbing the parameters with standard normal variates ξ. The values of the perturbations to k_1 and k_3 are $0{,}4\xi$ and $0{,}015\xi$.

THE DEVELOPMENT OF THE MODELS

These two models were developed by using non-linear least-squares methods to calculate the parameters of the model for each of the individual tests and then by using regression techniques to predict the parameter values over all of the tests. In this model building, frequent comparisons of the model predictions and the data are made, and regression techniques are used to test for, and identify, non-random behaviour in the error terms. As the models predict several data values, each of these values has been tested separately. The outline of the model-building procedure is:

(i) Propose form for model which includes unknown parameters.
(ii) Write a computer programme to simulate the unit using the proposed model. Test for correct implementation.
(iii) (a) Calculate the unknown parameters within each test by non-linear least-squares estimation.

or

(b) Calculate parameters over all tests by non-linear least-squares estimation and then proceed to step (vii).

(iv) Examine the closeness of fit in (iii) for unsatisfactory behaviour.

(v) Attempt prediction of the model parameters from (iii) (a) from the operating parameters using regression techniques.

(vi) Programme the prediction equations found in (v) into the model and test them for correct implementation.

(vii) Compare the model predictions with all the available data and test the error terms for any significant non-random behaviour.

(viii) Incorporate random number generators into the model to simulate the random behaviour of the difference between the model and the data. Test these additions again for correct implementation.

(ix) Compare the predicted errors with the actual errors for each test.

(x) Use other tests such as behaviour in conjunction with other models or predictions from internal sections of the model to increase the confidence in the model.

Should the results of any of the above comparisons be unsatisfactory, the model is revised at some level, and the above procedure continued from that level. The details of each step in the above procedure are given in Whiten (1971 a).

The regression technique used was multiple spline regression. This uses natural cubic spline functions, which are very smooth interpolation curves (Ahlberg, *et al*, 1967), to predict the dependent variable. In this technique, restraints on the amount of detail replace assumptions on the analytic form of the regression equation and the emphasis is on the response surface or its component response surfaces, rather than the coefficient in an analytic expression. Ordinary linear regression is obtained as the simplest case of the multiple spline regression which is generally more flexible and provides a more easily controlled tool than stepwise multiple regression analysis (Efroymson, 1960) when the analytic form of the relation being sought is not known. Multiple spline regression provides more information on the nature of the data and is less likely to impose an improper form onto the data. Details of this technique are given in Whiten (1971 b).

In the case of the screen and cone crusher models, it was found necessary to use only one one-dimensional non-linear response and hence the prediction equations for the model parameters contain only one spline function, the remaining terms being linear.

SIMULATION OF THE PRIMARY SECTION

The data obtained for the primary section were not considered extensive or reliable enough to build independent models for this section. The models of the screen and crusher constructed for the secondary section were found to give adequate prediction for the primary section. As was expected, the differences between the data and the model were more scattered, giving errors in the percentage sizing with a mean of 2,0 and a standard deviation of 3,5.

SIMULATION OF THE COMPLETE CRUSHING PLANT

For the simulation of the complete plant, the closed-circuit operation of the secondary section must be simulated. This is done by assuming a crusher discharge (initially, zero is adequate) and then simulating the units around the closed loop following the direction of ore flow so that an improved estimate of the crusher discharge is obtained. This improved estimate of the crusher discharge is used to enable the units around the loop to be simulated again and this process is repeated until it converges. A relaxation factor may be used to improve the rate of convergence. Normally 15 to 20 iterations are required to find the steady-state behaviour. As the simulation is quite fast, this presents no problems on a digital computer.

The test data for the tests on the complete plant were simulated. An acceptable reproduction of the test data was obtained. The secondary crusher current had a mean error of 2,3 amperes and a standard deviation of 3,2 amperes, and the crusher load a mean error of 1,3 tons/h and a standard error of 25 tons/h. The percentage sizings of the plant product had a standard error of 3,5. The data error prediction (by perturbing the parameters) of the model provided a reasonable reconstruction of the distribution of these errors.

DEDUCTIONS FROM THE SIMULATION OF THE COMPLETE CRUSHING PLANT

The simulation of the complete plant allows a precisely controlled series of simulated tests to be run. Such a series of tests on the actual plant would be very time-consuming and usually spoilt by random variations in the test conditions and data values. A series of tests involving a total of 21 plant simulations, which cost less than $10 for computer time, was run. The following variables were varied individually over their operating range:

(i) secondary crusher gap,

(ii) secondary screen size,

(iii) primary screen size,

(iv) primary crusher gap, and

(v) tons per hour of new feed.

Of these simulations, the last was of particular interest, as it showed that an increase of three amperes on the secondary crushers corresponded to an increase of 50 tons/h in plant throughput, with a slight decrease in product size at the higher feed rates. To take advantage of this, a control system was designed to remove some of the variations in the crusher current so that a smaller safety margin would be required, and hence a higher throughput could be obtained. This control system was also designed to ensure that either the primary section or the secondary section would run at a set maximum current. The balance between primary and secondary sections is controlled from the surge bin levels.

This control system has been installed at Mount Isa and has increased crushing capacity by 15 to 20 per cent.

The effect of size of crusher gap, which increases slowly during operation, was examined in another 36 simulations of the entire plant. The actual tonnage at which the plant would run under the control system was determined by interpolation in tables of the plant operation at various tonnages and fixed crusher gaps. Hence a table of the variations in product size and plant tonnage with primary and secondary crusher gaps for the control system was obtained (Fig. 4). These results also showed that optimum crushing plant operation occurs when both primary and secondary crusher currents are at their maximum values. This optimum is at the corners on the constant crusher gap lines on Fig. 4.

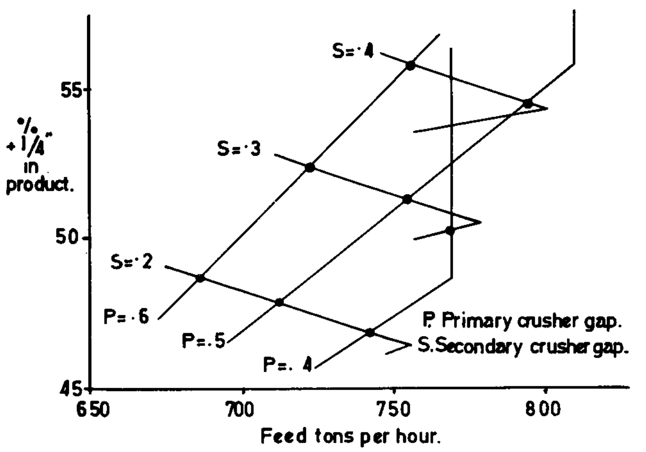

Fig. 4—Crushing plant operating curves under automatic control

CONCLUSION

These models of the cone crusher and vibrating screen, while certainly not beyond improvement, provide a reasonably comprehensive and accurate simulation of the Mount Isa Mines plant. Scale factors have been included in the models so that equipment of other sizes can be simulated. A set of less extensive data from the NBHC crushing plant at Broken Hill has been reproduced successfully by these simulation models and the effect of conversion of the plant to closed circuit calculated. Hence, it appears that these models can be applied to most crushing plants consisting of cone crushers and screens.

Once a suitable simulation is achieved, many tests on the simulated plant can be carried out at a very low cost. Further, the results of these simulations do not display the large amount of random variation that is typical of data from actual crushing plants. Hence the interpretation of the simulation results is much easier, and provides more definite conclusions than can be obtained from direct experiments on a crushing plant.

ACKNOWLEDGEMENTS

The support of Mount Isa Mines throughout the duration of this project is acknowledged. New Broken Hill Consolidated provided further data. Dr A. J. Lynch provided many helpful discussions during the development of these models.

REFERENCES

AHLBERG, J. H., NILSON, E. N., and WALSH, J. L. (1967). *The theory of splines and their applications.* Academic Press, New York.

BROADBENT, S. R. and CALLCOTT, T. G. (1956) *Coal Breakage Processes.* J. Inst. Fuel, vol. 29, pp. 529-539.

DRAPER, N., DREDGE, K. H. and LYNCH, A. J. (1969). Operating behaviour of an automatic control system for a mineral grinding circuit. *Ninth Commonwealth Mining and Metallurgical Congress.* Paper 22.

EFROYMSON, M. A. (1960). Multiple regression analysis. *Mathematical methods for digital computers.* vol. 1. By Ralston, A. and Wilf, H. S. (Eds.). Wiley, New York. pp. 191-203.

HART, J. *et al* (1968) *Computer Approximations* Wiley, New York.

LYNCH, A. J., WHITEN, W. J. and DRAPER, N. (1967). Developing the optimum performance of a multi-stage grinding circuit. *Trans. Instn. Min. Metall.* vol. 76. pp. C169-C182.

WHITEN, W. J. (1971 a). Model building techniques applied to mineral treatment processes. *Austr. Inst. Min. Metall. (Southern Queensland Branch) Symposium in Automatic Control Systems in Mineral Processing Plants.* Paper 8.

WHITEN, W. J. (1971 b). The use of multi-dimensional cubic spline functions for regression and smoothing. *Austr. Comp. J.* vol. 3, No. 2. pp. 81-88.

APPENDIX A

A cone crusher algorithm

Notation

A	amperage
$b_1\ b_2\ b_3$	scaling factors
b'_i	coarse breakage function
b''_i	fine breakage function
f_i	components of feed vector
g	crusher gap
$k_1\ k_2\ a$	model parameters
n	number of size fractions
p_i	components of product vector
q	fraction plus one inch in feed

s_i	upper size limit of i-th fraction
t	tons per hour of feed
t_i	a temporary vector
$SPNV$	interpolating spline function

1. Calculation of breakage function

a. Relative size (coarse) breakage

1.1 $w=\sqrt{s_1 s_2}$; $u=1-e^{-1}$; $v_2=1$

2.1 $v_1=\{1-exp[-(s_{i+1}/w)^6]/u\}$

2.2 $b_i'=v_2-v_1$ — for $i=1$ to $n-1$

2.3 $v_2=v_1$

b. Fines production

1.1 $v_2=1$

2.1 $v_1=exp\ (-s_{i+1}/0{,}12)^{1{,}25}$

2.2 $b_i''=v_1-v_2$ — for $i=1$ to $n-1$

2.3 $v_2=v_1$

2. Model parameters

1.1 $k_1=0{,}67g$

1.2 $k_2=1{,}121g+2{,}31q\ b_1+b_1\ SPNV$ ([100.,250.,400.],[$-0{,}047\ 7$, $-0{,}084\ 9$, $-0{,}259\ 3$], t/b_1)

1.3 $a\ =0{,}8723+0{,}115g$

3. Calculation of product

Define function $F(s)$ as

$$
\begin{aligned}
F(s) &= k_1+\frac{1}{3}(k_1-k_2) && k_1>s\\
&= s+\frac{1}{3}\frac{(s-k_2)^3}{(k_1-k_2)^2} && k_2>s>k_1\\
&= s && s>k_2
\end{aligned}
$$

1.1 $c\ =[F(s_j)-F(s_{j+1})]/(s_j-s_{j+1})$

1.2 $$x=\frac{f_j-\sum_{i=1}^{j-1}[ab_j'{}_{-i+1}+(1-a)b'']t_i}{1-[ab_1'+(1-a)\sum_{i=1}^{j}b_i'']c}$$ — for $j=1$ to $n-1$

1.3 $t_j\ =cx$

1.4 $p_j=x-t_j$

2.1 $p_n=\sum_{i=1}^{n} f_i-\sum_{i=1}^{n-1} p_i$

4. Calculation of amperage

1.1 $a\ =\sum_{i}^{n} t_i/(s_i+s_{i+1})$

1.2 $A\ =b_2+b_3\ (0{,}082\ 2a+0{,}000\ 305a^2/b_1)$

APPENDIX B

The vibrating screen algorithm

Notation

c_i	coarse product vector components
d	wire diameter
f_i	feed vector components
h	hole size
$k_1\ k_2\ k_3$	model parameters
l	length of screen
n	number of size fractions
p_i	fine product vector components
s_i	upper size limit of i-th fraction
v	volumetric feed rate
w	width of screen

1.1 $$Z = k_1\sqrt{\frac{l}{1+(k_2 v/wh)^2}}\ /\ (h+d)$$

Define $g(x) = \frac{0{,}124\ 734}{Z} / \{[(x-0{,}437\ 880\ 5)x+0{,}266\ 982]x+0{,}138\ 375\}$

2.1 $c_i = f_i$

2.2 $p_i = 0$

for i such that $s_{i+1} > h$

3.1 $$r = \frac{s_i - h}{s_i - s_{i+1}}$$

3.2 $$c_i = f_i\left(r+(1-r)\frac{0{,}89/Z - g[Z(h-s_{i+1})]}{h-s_{i+1}}\right)$$

3.3 $p_i = f_i - c_i$

for i such that $s_i > h > s_{i+1}$

4.1 $$c_i = \frac{g[Z(h-s_i)] - g[Z(h-s_{i+1})]}{s_i - s_{i+1}}$$

4.2 $p_i = f_i - c_i$

for i such that $h > s_i$ and $i \approx n$

5.1 $c_n = 0{,}1 f_n$

5.2 $p_n = f_n - c_n$