Weakness correlation and the size effect in rock strength tests

J. P. G. PRETORIUS*, M.Sc. (Pret.)

SYNOPSIS

The size effect in tests of the compressive strength of coal specimens differs from predictions based on the Weakest Link theory. Predictions of the size effect in tensile tests on lengths of cotton yarn and glass fibre, the behaviour of which is analogous to the weak link process, also differ from observations. This discrepancy is shown to be caused by the correlation between the strengths of the unit lengths in a composite specimen. The Weakest Link theory is reformulated as the Weakness Correlation theory to account for such correlation. It is shown that weakness correlation is found in rock and it is concluded that at least part of the discrepancy between the predicted and observed size effect in rock strength tests could be attributed to this correlation.

SINOPSIS

Die waargenome grootte effek in druksterkte toetse op steenkool monsters verskil van voorspellings gebaseer op die Swakste Skakel teorie. Voorspellings van die grootte effek in treksterkte toetse op katoengaring en glasvesels waar die gedrag ooreenkom met die swakste skakel proses, verskil ook van waarnemings. Dit word aangetoon dat hierdie verskille toegeskryf kan word aan die korrelasie tussen die sterktes van eenheid lengtes in 'n saamgestelde monster. Die Swakste Skakel teorie word herformuleer as die Swakheids Korrelasie teorie om rekening te hou met hierdie korrelasie. Verder word aangetoon dat swakheid korrelasie in rots voorkom en die gevolgtrekking word gemaak dat tenminste 'n gedeelte van die verskil tussen die voorspelde en waargenome grootte effek in swigtingstoetse op rots hieraan toegeskryf kan word.

INTRODUCTION

Estimates of the strength of rock structures, such as coal pillars, are usually based on the results of compression tests on small specimens. The results of such tests vary considerably and cannot be used directly to estimate the strength of large volumes of the rock, since these are weaker than smaller volumes.

The 'Weakest Link' theory has been used extensively to account for this size effect. Evans and Pomeroy[1], Skinner[2] and Grobbelaar[3], amongst others, used the theory in analyses of the results of compression tests on coal, anhydrite, norite and concrete specimens. In this paper the validity of the use of this theory is considered.

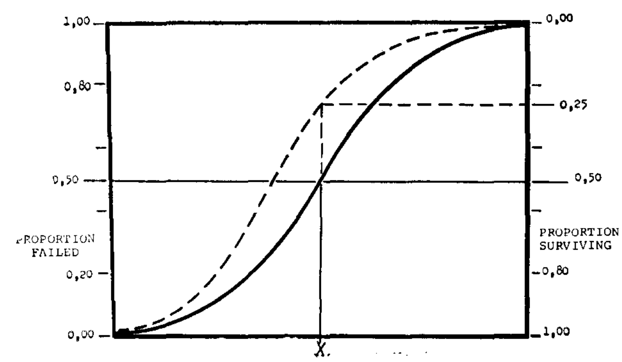

Fig. 1—Cumulative strength distributions
Single length ————
Double length — — — — —

THE WEAKEST LINK THEORY

Although Pierce[4] was the first to formulate the Weakest Link theory quantitatively, the general acceptance of the theory resulted mainly from the publication of Weibull's[5] classical paper. His derivation of the basic part of the theory is as follows:

If a large number of glass rods n with a given length l (l can be any arbitrary length) and a constant diameter were to be tested in tension, these would rupture at different intensities of load. The number of specimens m which have failed at load x, or less than x, would be known and the proportion m/n could be plotted against the load as shown in Fig 1 (solid curve).

It will be seen from the graph in Fig. 1 that a proportion of 0,5 of the hypothetical specimens survived a load x. If a large number of similar glass rods of length $2l$ were to be tested, the probability that the upper half (length$=l$) would survive a load x would be 0,5. Similarly the survival probability of the lower half at this stress would be 0,5. For the rod as a whole to survive, both halves should be intact, the probability of this eventuality being $0{,}5 \times 0{,}5 = 0{,}25$. In general if the probability of failure of a unit length at (or less than) x is $F_1(x)$, the probability of survival is $(1-F_1(x))$ and the probability of survival of a specimen consisting of n unit lengths, $(1-F_n(x))$, is given by the following equation:

$$1-F_n(x) = (1-F_1(x))^n \quad \text{(i)}$$

In the derivation of equation (i) two assumptions were made. The first (entirely reasonable) assumption is that $F_1(x)$ exists. The second is that the strengths of the unit lengths in a composite specimen are independent. This assumption has never, to the author's knowledge, been stated explicitly and it is clear from the publications of, amongst others, Weibull[5] and Kontorova and Frenkel[6] that it was not realised by workers in this field that this assumption was implicit in their formulation of the theory. Even the comprehensive study by Epstein[7] of the theory dealt only with the case where the strengths of the 'links' were independent.

*Senior Research Officer, Dept. of Mining Engineering, University of the Witwatersrand

THE NATURE AND STRENGTH FREQUENCY DISTRIBUTION OF THE 'LINK'

A unit volume or length as a link.

In Weibull's derivation of equation (i) the unit length is equivalent to a 'link' and a specimen of length $n \times l$ (n integer) equivalent to a chain of n such links. If n is a fraction p/q, the unit length can be regarded as consisting of q sublengths. If the cumulative failure distribution of those sublengths is $F_s(x)$, it follows that

$$1-F_1(x) = (1-F_s(x))^q$$

and $1-F_n(x) = (1-F_s(x))^p$

$$\therefore 1-F_n(x) = (1-F_1(x))^{p/q} \quad \ldots\ldots\ldots \text{(ii)}$$

Regardless of how small these sublengths are they are considered to be continuous and no reference is made to flaws in the material.

Although the theory was derived for tests on thin rods in tension, it was assumed that the 'chain of links' analogy could also be used for compression tests on solids. In this case the ratio p/q would refer to the volume ratio of the specimens being compared.

Weibull obtained a frequency function for the strength of the links as follows:

From equation (i)

$$\log(1-F_n(x)) = n\log(1-F_1(x)) = n\,g(x) \quad \ldots\ldots\ldots \text{(iii)}$$

when $x = 0$, $F_1(x) = 0$ and $g(x) = 0$

when $x \rightarrow \infty$, $F_1(x) \rightarrow 1$ and $g(x) \rightarrow -\infty$

$F_1(x)$ is monotonically non-decreasing, therefore $g(x)$ must be monotonically non-increasing. Weibull arbitrarily chose the following function:

$g(x) = -(x/a)^b$

It will be seen that the function meets the requirements outlined above. From equation (iii) it follows that

$$F_n(x) = 1-e^{ng(x)}$$

$$= 1-e^{-n(x/a)b} \quad \ldots\ldots\ldots\ldots \text{(iv)}$$

The flaw as a link

Kontorova and Frenkel[6] regarded each flaw in a solid as analogous to a link. The strength of each such link would be that of the equivalent flaw and should apparently be interpreted as being the stress at which the flaw would start to propagate. If the analogy between the strength a chain of such links and that of a solid is to be used to determine the frequency distribution of the strength of the solid, the following must be known:

(i) The frequency distribution of the total number of flaws, since the number of flaws will vary between specimens. The length of the hypothetical chain will, therefore, also vary.

(ii) The frequency distribution of the strengths of such flaws.

(iii) The association between the number (in (i)) and strengths (in (ii)), since specimens with a larger number of flaws might have smaller flaws than those with a smaller number of flaws.

(iv) The interaction between flaws. In the process of rupturing flaws will link. The resultant flaws will have a changed strength frequency distribution and the length of the chain will shorten.

In view of our ignorance about the factors mentioned in paragraphs (i) to (iv), the simple assumption by Kontorova and Frenkel that flaw strength has a Gaussian distribution and that of Checulin[8] that it has a Gamma distribution is not justified and, at present, impossible to verify.

A CRITERION FOR TESTS OF THE VALIDITY OF THE WEAKEST LINK THEORY

It is clear from the foregoing discussion that experimental verification of the Weakest Link theory based on the flaw as a link is not feasible. The fact that Weibull's distribution can be fitted to strength frequencies also does not constitute proof of the validity of the theory, since his choice of $g(x)$ was quite arbitrary.

In most applications of the Weakest Link theory a measure of central tendency of the strength frequency distribution, such as the mean, median or mode is plotted against the specimen size and the type of curve predicted by theory fitted to it. Two or three parameters are estimated from the data and it is not surprising that a good fit is usually obtained.

To evaluate the theory an experiment should be conducted in such a way that the parameters describing the strength frequency distribution can be estimated a priori. One way in which this can be done is to conduct a number of strength tests on specimens (of the same material) of two sizes. The proportion of specimens of the smaller size which have failed at a given load x, or less, provides an estimate of $F_1(x)$. An estimate of $F_n(x)$ can be obtained in the same manner for the larger specimens. In this estimation use is not made of the properties of any assumed type of frequency distribution. In order to have reliable estimates of $F_1(x)$ and $F_n(x)$ at a number of values of x, a large number of tests should be conducted.

THE STRENGTH OF COAL CUBES

Evans and Pomeroy[1] published results of a large number of compression tests (between rigid steel platens) on Deep Duffryn and Barnsley Hard coal cubes of different sizes. Under these test conditions the stress would not be the same throughout the specimen. The results are, however, comparable, since given proportions of the volume of the specimens are subject to the same stress range.

It was stated earlier that a large number of tests are required to obtain estimates of $F_1(x)$ and $F_n(x)$. This part of the analysis is, therefore, limited to a comparison between specimens of Deep Duffryn coal with sides 5,84 mm (262 specimens) and 12,70 mm (164 specimens) as well as Barnsley Hard coal specimens with sides 3,18 mm (157 specimens) and 6,35 mm (445 specimens). According to the Weakest Link theory $n=p/q$ in equation (ii) should in this case be the ratio of volumes. For the Deep Duffryn specimens this proportion is, therefore, 10,3 and for the Barnsley Hards 8,0.

In Tables 1(a) and 1(b) the observed frequencies and those expected on the basis of the theory is given. The intervals were chosen to contain at least 10 observations or predicted values at all strength levels for the three columns.

TABLE I

FREQUENCIES OF STRENGTH OF COAL CUBES OBSERVED AND PREDICTED BY THE WEAKEST LINK THEORY (BASED ON VOLUME RATIO)

I (a) Deep Duffryn Coal

Strength (Bars)	Cube side length (mm)		
	5,84	12,70	
		Observed	Predicted
Less than 23200	10	14	54
23200-29000	12	11	43
29000-34800	18	27	37
34800-37700	18	11	17
More than 37700	204	101	13

$$\chi^2_4 = 683 \qquad P < .005$$

I (b) Barnsley Hard Coal

Strength (Bars)	Cube side length (mm)		
	3,18	6,35	
		Observed	Predicted
Less than 123300	12	41	209
123300-145000	17	83	149
145000-152300	12	48	47
More than 152300	116	273	40

$$\chi^2_3 = 1543 \qquad P < .005$$

It is clear from the results given in Table I that the predictions based on the Weakest Link theory differ markedly from the experimental observations. The theory predicts a larger number of weak and a smaller number of strong specimens than observed. The results of the χ^2 tests show that this difference is highly significant.

Evans and Pomeroy plotted the mean strengths (on a logarithmic scale) against the lengths of the sides of the cubes and fitted a straight line to these data. On the basis of the resultant fit they came to the conclusion that the ratio n in equation (ii) should be the ratio between the side lengths and not the volumes of the specimens.

In Tables II(a) and II(b) the observed frequencies and those predicted by the Weakest Link theory with n taken as the side length ratios are given. Intervals were again chosen to include a minimum of 10 observations or predicted values.

It will be seen from the results in Table II that the predictions based on this modification of the theory are considerably better than those given in Table I. However, they differ from the observed frequencies and the χ^2 tests show that this difference is significant.

The results of analyses based on other specimen sizes confirms the conclusion that the Weakest Link theory, either with n in equation (ii) as the volume ratio, or with n as the side ratio, cannot be used to account for the size effect observed. These results are, however, not

TABLE II

FREQUENCIES OF STRENGTH OF COAL CUBES OBSERVED AND PREDICTED BY THE WEAKEST LINK THEORY (BASED ON SIDE LENGTH RATIO)

II (a) Deep Duffryn Coal

Strength (Bars)	Cube side length (mm)		
	5,84	12,70	
		Observed	Predicted
Less than 23 206	10	14	13.3
23 206 29 010	12	11	15.2
29 010 34 810	18	27	21.1
34 810 37 710	18	11	19.2
37 710 43 510	26	33	24.4
43 510 52 210	41	39	30.7
52 210 58 010	32	13	17.4
More than 58 010	105	16	22.4

$$\chi^2_7 = 14.6 \qquad P < .05$$

II (b) Barnsley Hard Coal

Strength (Bars)	Cube side length (mm)		
	3,18	6,35	
		Observed	Predicted
Less than 123 280	12	41	65.4
123 280-145 040	17	83	83.8
145 040-152 290	12	48	52.9
152 290-159 540	12	49	47.7
159 540-174 050	25	111	82.6
174 050-181 300	16	46	41.0
181 300-188 550	10	32	20.9
188 550-195 800	21	21	32.2
More than 195 800	32	14	18.5

$$\chi^2_8 = 30.8 \qquad P < .005$$

given here, since the following analyses will give further support to the conclusion.

The possibility was investigated that a value of n other than the volume or side length ratio might satisfy equation (ii). Using the data of Evans and Pomeroy, estimates of $F_1(x)$ and $F_n(x)$ were obtained at different strength values and n estimated. The values of n obtained in this manner are plotted against the strength in the graphs in Figs. 2 and 3.

It is evident from the graphs in Figs. 2 and 3 that the values of n vary considerably and show a tendency to increase with an increase in strength.

A summary of the correlations found between n and strength is contained in Table III.

In all cases considered in Table III the correlations were found to be positive. The Rank Correlation coefficient was used to determine if the correlations were significant since it is distribution free.

In three of the cases the correlations were found to be significant at a 5 percent level. It is clear, therefore,

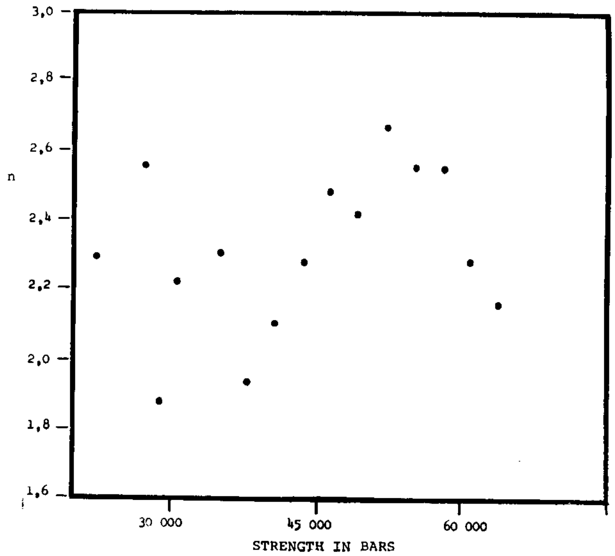

Fig. 2—Values of *n* in equation (ii) for Deep Duffryn 5,84 and 12,7 mm cubes

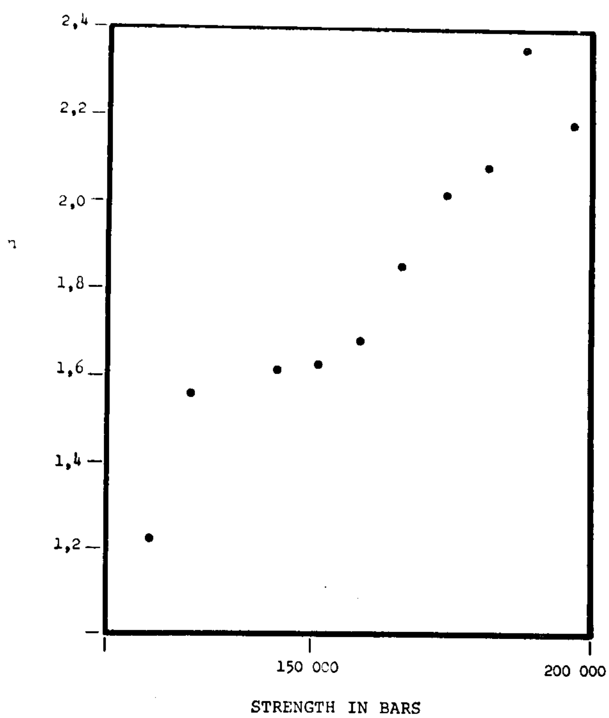

Fig. 3—Values of *n* in equation (ii) for Barnsley Hard 3,18 and 6,35 mm cubes

that no single value of n in equation (ii) will suffice to represent the relationship between the strength frequency distributions.

TENSILE TESTS ON LENGTHS OF COTTON YARN AND GLASS FIBRE

The fact that the Weakest Link theory in its present form cannot be used to describe the size effect in rock strength tests can be attributed to one or both of the following causes.

(i) The rupture process does not bear any formal relationship to the failure of a single chain of links.

TABLE III

VALUES OF n (IN EQUATION (ii))

Comparison between	Correlation between n and strength	
	Product moment correlation coeficients	Kendall rank correlation coeficients
Deep Duffryn		
5,84 and 12,70 mm	0,31	0,19
12,70 and 23,64 mm	0,75	0,60
5,84 and 23,62 mm	0,92	0,87*
Barnsley Hards		
3,18 and 6,35 mm	0,95	0,96*
6,35 and 25,40 mm	0,99	1,00*
3,18 and 25,40 mm	0,25	0,06

*Indicates that rank correlation is significant at 5% level.

(ii) The formulation of the Weakest Link theory does not describe the process of rupture even when the chain analogy is applicable.

The latter possibility can be investigated by testing the validity of the theory in rupture processes directly analogous to a chain of links. The tests conducted by Pierce[4] and Kapadia[9] on cotton yarns and those by Kies[10] on glass fibres in tension can be regarded as being closely analogous to chains with a different number of links. If a thin specimen in tension consists of any number of unit lengths, failure of any one of these lengths will be equivalent to the failure of the composite specimen.

Pierce reported the results of tests on 200 specimens each of 10 and 30 cm lengths of cotton yarn. The value of n in this case is, therefore, 3 and the observed and predicted frequencies are given in Table IV.

TABLE IV

FREQUENCIES OF THE STRENGTH OF COTTON YARNS OBSERVED AND PREDICTED BY THE WEAKEST LINK THEORY (PIERCE'S DATA)

Load (gm)	Length of yarn (cm)		
	10	30	
		Observed	Predicted
Less than 170	13	17	36.5
170 180	20	27	47.0
180 190	18	38	33.7
190 200	29	33	39.5
200 210	24	29	21.1
210 220	23	24	12.4
More than 220	73	32	7.7

$$\chi^2_6 = 85 \qquad P < .005$$

From the results given in Table IV it will be seen that the Weakest Link theory predicts more weak and fewer strong specimens than observed. The χ^2 test shows that this difference is significant.

The results of analyses on thousands of specimens of cotton yarn reported by Kapadia[9] confirm this conclusion without exception, as do the tests on glass fibres reported by Kies.

REFORMULATION OF THE WEAKEST LINK THEORY

Since the results of tests on cotton yarns and glass fibres in tension which are closely analogous to a weak link process do not agree with the predictions based on the Weakest Link theory its formulation must be reconsidered.

Using Weibull's example in Section 2 and considering the case where the specimen consists of two unit lengths, we note that each of these lengths have a frequency density distribution $f(x)dx$. For the specimen as a whole to survive both unit lengths must survive. The probability of this eventuality is as follows:

$$1-F_2(x)=\int_x^\infty \int_x^\infty f(x_1, x_2)dx_1\, dx_2 \quad \ldots\ldots\ldots \quad \text{(v)}$$

where $f(x_1, x_2)dx_1, dx_2$ is the bivariate strength frequency density function of the two unit lengths.

If there is no correlation between x_1 and x_2 equation (v) is equivalent to the following:

$1-F_2(x) = (1-F_1(x))^2$

which is in the form of equation (ii).

If there is positive correlation between x_1 and x_2 then

$1-F_2(x) > (1-F_1(x))^2$

In general, if a specimen consists of n unit lengths

$$1-F_n(x) = \int_x^\infty \int_x^\infty \ldots\ldots \int_x^\infty f(x_1, x_2 \ldots x_n)\, dx_1\, dx_2 \ldots dx_n \quad \ldots\ldots \quad \text{(vi)}$$

If there is positive correlation between the n unit lengths

$$1-F_n(x) > (1-F_1(x))^n \quad \ldots\ldots\ldots\ldots \quad \text{(vii)}$$

In all the cases observed above the relationship given in equation (vii) was found to hold true.

Only two assumptions were made in the derivation of equation (ii). The first undisputed assumption is that $F_1(x)$ exists. The second assumption, namely that the strengths of the constituent unit lengths are independent, must, therefore, be incorrect and it follows that the strengths of these lengths are correlated.

Since the name 'Weakest Link theory' has been associated for close on half a century with the implicit assumption of independence of the strengths of unit lengths (or volumes) in a composite specimen, it is suggested that the new formulation of the theory given in equation (vi) should be referred to as the 'Weakness Correlation' theory.

It will be noted that this 'Weakness Correlation' theory reduces to the Weakest Link theory when the strength of the unit lengths are independent. On the other hand, if the correlation is equal to 1 it follows that each unit length has the same strength as the one preceding it. There will, therefore, be no variation or size effect and the theory will reduce to the 'classical' theory. The Weakness Correlation theory, therefore, bridges the gap beiween the classical and Weakest Link theories in rupture processes which are analogous to a chain of links in tension.

WEAKNESS CORRELATION IN ROCK

It may be contended that the weakness correlation observed in cotton yarns and glass fibres could be a result of varying diameter, with the thicker parts tending to occur together and vice versa. The evidence of weakness correlation in rock must, therefore, be examined.

Wiid[11] conducted uniaxial and triaxial tests on 299 and 100 specimens taken from a volume of sandstone less than 1 m cube. The exact location of each of the specimens was noted and it was possible, therefore, to determine the correlation between successive specimens in any drill hole. The Product Moment Coefficients were found to be 0,35 and 0,39 respectively, while the Kendall Rank Correlation Coefficients were 0,34 and 0,32. The former, based on 99 pairs of values is significant at a 5 percent level. The latter, based on only 10 pairs was not significant.

Kostak[22] tested 420 specimens of Elliot Lake Quartzite in compression. The specimens came from an area approximately 13 m by 2 m and the drill holes were approximately 5 m deep. The specimens had a diameter of approximately 5 cm and were on the average about 20 cm apart. The Product Moment Correlation Coefficient and Rank Correlation Coefficient were 0,2 and 0,15 respectively. The latter value is significant at a 5 percent level.

ESTIMATES OF THE CORRELATION IN SPECIMENS

If it assumed that equation (vi) can be used to relate the results given by Evans and Pomeroy, the correlation can be estimated by means of Hojo Integrals. The assumption implicit in the use of these integrals is that the correlations between all unit volumes in a specimen are the same.

The Product Moment Correlation Coefficients were found to be 0,66 and 0,68 for Deep Duffryn coal 5,84 mm and 12,7 mm specimens, while that for Barnsley Hard 3,18 mm specimens was 0,78.

These correlations are considerably higher than those in the previous test. It should be noted, however, that these specimens came from a large area, while the previous ones were taken out of relatively small volumes. The association between the strengths of adjacent parts of a composite specimen can, therefore, be expected to be higher, since it is measured against the variation over a large area. Furthermore, small changes in test conditions introduce variation in results which are additional to the material variability. This additional variation will cause the observed correlation to be lower than the true value.

CONCLUSIONS

It has been shown that the assumption that the strengths of the hypothetical 'links' in the chain (referred to in the Weakest Link theory) are independent is incorrect. It has furthermore been shown that there is evidence of correlation between the strengths of adjacent volumes in rock. The proposed Weakness Correlation theory can be used to account for this dependence in processes analogous to the rupture of a chain of links. Further experiments are required to determine if this will be sufficient to account for the size effect in rock strength testing. It seems reasonable, however, to conclude that at least part of the discrepancy between the theoretical predictions based upon the Weakest Link theory and experimental observations can be attributed to weakness correlation in rock.

REFERENCES

1. Evans, I. and Pomeroy, C. D. Mechanical Properties of Non-Metallic Brittle Materials, Butterworths, London, 1958.
2. Skinner, W. J. 'Experiments on the compressive strength of Anhydrite'. *Engineer*, Vol. 207, February, 1959.
3. Grobbelaar, C. 'The theoretical strength of mine pillars'. Part 1. *J. S. Afr. Inst. Min. Met.*, Vol. 69, November, 1968.
4. Pierce, F. T. 'Tensile tests for cotton yarns. (v). The Weakest Link'. *J. Tex. Inst.*, Vol. 17, 1926.
5. Weibull, W. 'A statistical theory of the strength of materials'. *Ingvetensk Akad, Handl.*, No. 151, 1939.
6. Kontorova, T. A. and Frenkel, J. I. 'A statistical theory of the brittle strength of real crystals'. *J. Phys.*, U.S.S.R., Vol. 7, 1943.
7. Epstein, B. 'Statistical aspects of fracture problems'. *J. Appl. Phys.*, Vol. 19, February, 1948.
8. Checulin, B. B. 'On the statistical theory of brittle strength'. *J. Tech. Phys.*, U.S.S.R., Vol. 24, No. 2, 1954.
9. Kapadia, D. F. 'Single-thread strength of yarns in various lengths of test specimens'. *J. Tex. Inst.*, Vol. 26, 1935.
10. Kies, J. A. 'The strength of glass'. N.R.L., Report 5098, April, 1958.
11. Wiid, B. L. 'The influence of moisture upon the strength behaviour of rock'. Ph.D. Thesis, University of the Witwatersrand, 1968.
12. Kostak, B. 'Strength distribution in hard rock'. *Int. J. Rock Mech. Min. Sci.*, Vol. 8, No. 5, September, 1971.