An Analytical Solution to the Batch-comminution Equation

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SYNOPSIS

An analytical solution to the batch-grinding equation is presented in terms of the generalized hypergeometric series and the confluent hypergeometric function. This solution is valid for selection functions that are described piece-wise by functional forms of the type k_0x^β and by normalized breakage functions of the type $B(x,x') = (x/x')^\alpha$. These functional forms are sufficiently general to include most practical cases. In particular, they can be used to describe selection functions that have a maximum.

SINOPSIS

'n Analitiese oplossing tot die enkel-vergruising vergelyking word aangegee in terme van die veralgemene hipergeometriese serie en die samevloeiende hipergeometriese funksie. Hierdie oplossing is geldig vir versamelings funksies wat stuksgewys beskrywe word deur funksionele vorme van die tipe k_0x^β en deur genormaliseerde opbreekbare funksies van die tipe $B(x,x') = (x/x')^\alpha$. Hierdie funksionele vorme is algemeen genoeg om meeste praktiese gevalle in te sluit. Hulle kan in besonder gebruik word om versamelings funksies te beskrywe wat 'n maksimum het.

INTRODUCTION

The fundamental integro-differential equation that describes the batch-grinding process has been known for some time. This equation,

$$\frac{\partial F\,(x,t)}{\partial t}=\int_x^{x_m} k\,(x')\,B\,(x,x')\,f\,(x',t)\,dx', \quad . \quad . \quad . \quad . \quad (1)$$

represents the mass balance over that fraction of the particles in a batch-grinding machine that are smaller than a characteristic size x. In equation (1), $F\,(x,t)$ is the fraction of material of size less than x. Because of the finite population of particles in the grinding machine, this function is necessarily discontinuous. However, because the particle population is large, the discontinuities are presumed to be small enough to be ignored and $F\,(x,t)$ is a continuous approximation to the fraction of particles. On this basis, $F\,(x,t)$ is differentiable, and $f\,(x,t)$ is defined as $f\,(x,t)=\partial F\,(x,t)/\partial x$. A precise interpretation of $F\,(x,t)$ can be given as the expected value of the number of particles of size less than x. Such an interpretation is the real justification for the assumption that $F\,(x,t)$ is differentiable and it leads to equation (1). The simple interpretation is adopted here because of its familiarity to readers of this Journal. The interested reader can consult the excellent paper by Filippov[1] for the alternative method. The initial charge can contain significant quantities of closely sized material, and it is then convenient to allow $F\,(x,t)$ to have jump discontinuities at these sizes. $F\,(x,t)$ will then contain generalized functions of Dirac delta-function type.

$B\,(x,x')$ is the fraction of material having size smaller than x that results from primary breakage of particles of size x', and again the small discontinuities are smoothed and $B\,(x,x')$ is presumed to be continuous and differentiable.

$k\,(x)$ is the specific rate for the breakage of particles of size x, and, as indicated, it is a function of the particle size for physical reasons. $k\,(x)$ is called the selection function.

Because there is no limitation on the functional form of $k\,(x)$ and $B\,(x,x')$ when numerical methods are used, it is customary to generate numerical solutions to equation (1). In addition, it is a comparatively simple matter for a machine to be programmed to generate a solution. However, analytical solutions are useful for a number of reasons. They provide a check on the accuracy of the numerical methods, and such a check is essential when a numerical procedure is used. Perhaps of more importance is the fact that they provide a basis for the efficient estimation of the breakage parameters from experimental batch-grinding data. The recent critical study of parameter estimation by Austin and Bhatia[2] has shown that, in the absence of efficient parameter-estimation techniques, lengthy special-purpose experiments are required. An analytical solution to equation (1) enables data obtained from normal batch-grinding tests to be analysed by conventional non-linear regression techniques. A further advantage of the analytical method is that it permits solutions of the comminution equation to be obtained by hand computation, whereas the numerical solution is not feasible without the use of a computer. The analytical solution was obtained in terms of various hypergeometric series, and to facilitate manual computation numerical values of these series are plotted in Figures 1 to 7.

A few analytical solutions have been obtained for highly restricted functions $k\,(x)$ and $B\,(x,x')$ and are reported in the literature. Gaudin and Meloy[3] apparently reported the first analytical solution with $k\,(x)=k_0x$ and $B\,(x,x')=x/x'$. These simple forms do not correspond to those based on experimental data, and their solution has not found much practical application. Bass[4] obtained a formal analytical solution to equation (1) in terms of a quadrature involving an infinite series of kernels that are evaluated iteratively, but this solution is by no means convenient to use. Recently Austin[5] summarized most of the known

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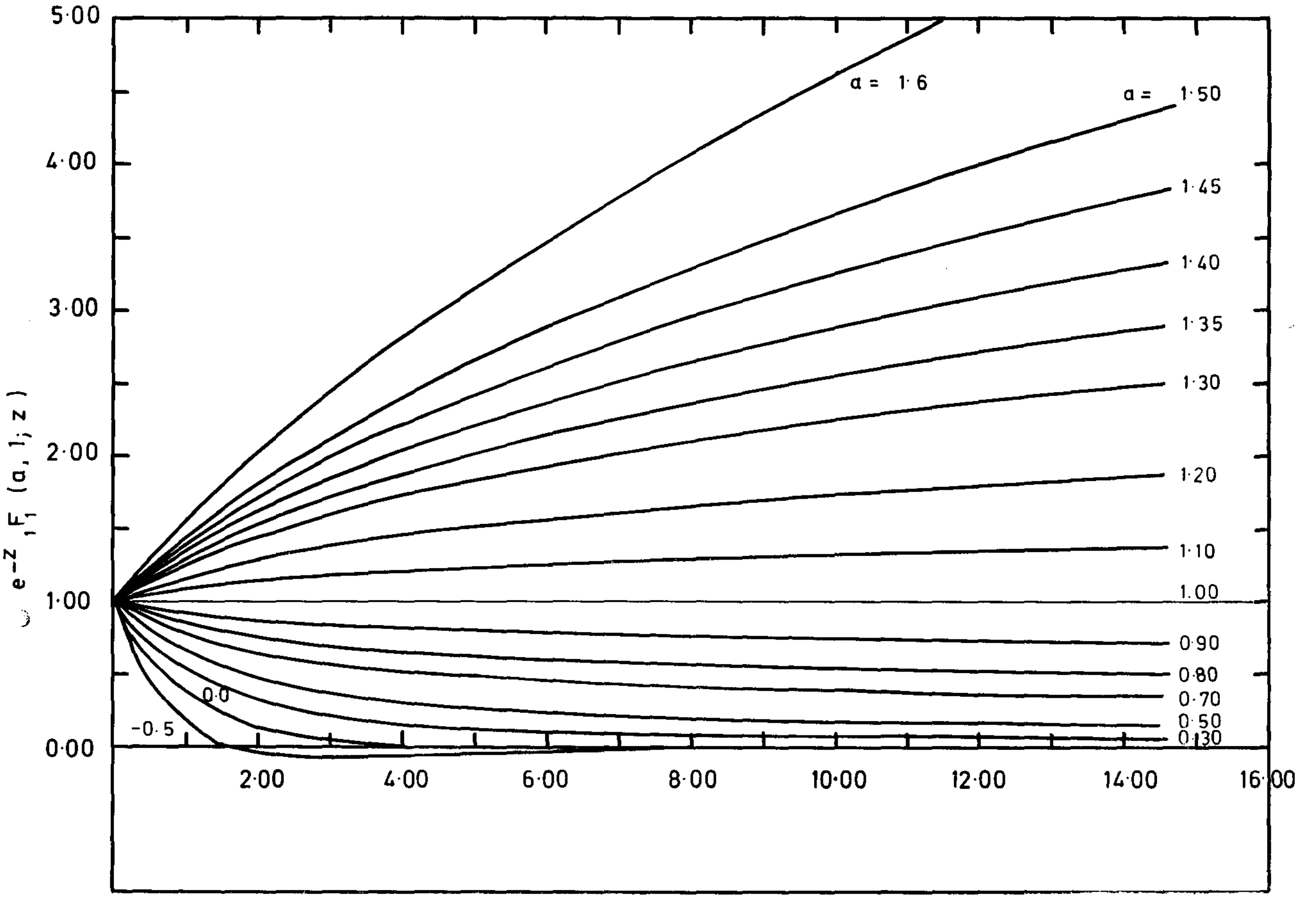

Fig. 1 The Confluent Hypergeometric Function $_1F_1$ (a,1;z) as a Function of a and z

analytical solutions and extended these to more realistic forms for the selection and breakage functions. In particular, he obtained approximate solutions for functional forms of the type described in this paper. Austin neglects the important work of Filippov, whose solutions are special cases of the type obtained here. Loveday[6] has explored an approximate solution in which he assumes that the solution, $f(x,t)$, is always a log-normal distribution. This approach promises to be very useful for the estimation of the selection and breakage functions from laboratory data. Analytical solutions of equation (1) are useful for the assessment of the errors inherent in the approximate solutions of Austin and Loveday.

THE ANALYTICAL SOLUTION

It is assumed that $B(x,x')$ can be written in the form

$$B(x,x') = \frac{B_1(x)}{B_1(x')} \quad \ldots \quad (2)$$

Such a form satisfies the essential requirement $B(x,x)=1$. By differentiation of equation (1),

$$\frac{\partial f(x,t)}{\partial t} = \frac{B_1'(x)}{B_1(x)} \int_x^{x_m} k(x')\, B(x,x')\, f(x',t)\, dx' - k(x) f(x,t),$$

where $B'_1(x) = \dfrac{dB_1(x)}{dx}$. $\quad \ldots \quad (3)$

From equation (3)

$$\int_x^{x_m} k(x')\, B(x,x')\, f(x',t)\, dx'$$

$$= \frac{B_1(x)}{B_1'(x)} \left[\frac{\partial f(x,t)}{\partial t} + k(x) f(x,t)\right],$$

and this integral can be substituted into equation (1) to give

$$\frac{\partial F(x,t)}{\partial t} = \frac{B_1(x)}{B'_1(x)} \left[\frac{\partial f(x,t)}{\partial t} + k(x) f(x,t)\right]. \quad \ldots \quad (5)$$

It can be seen therefore that, under the restriction of equation (2), equation (1) can be converted to a partial differential equation — equation (5).

Equation (5) is linear in the time, and the Laplace transform is useful. If the Laplace transform is defined by

$$\bar{F}(x,s) = \int_0^{\infty} e^{-st} F(x,t)\, dt, \quad \ldots \quad (6)$$

equation (5) transforms to

$$\frac{d\bar{F}(X,s)}{dx} - \frac{sB'_1(x)\, \bar{F}(x,s)}{B_1(x)\,(s+k(x))} = \frac{f(x,o)}{s+k(x)}$$

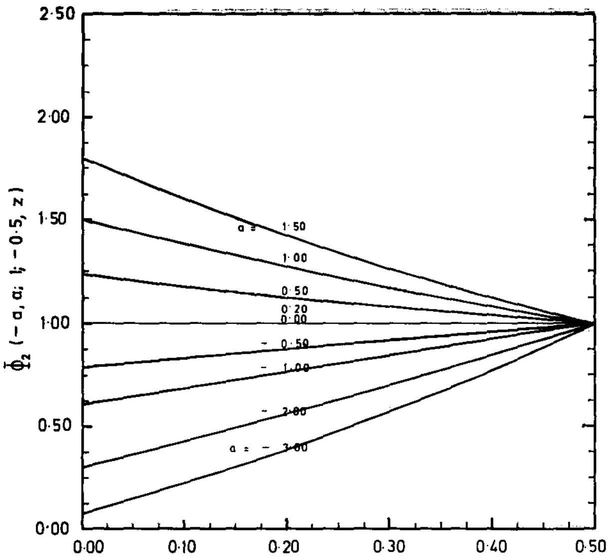

Fig. 2 The Hypergeometric Series Φ_2 $(-a,a,1;-0.5,z)$ as a Function of a and z

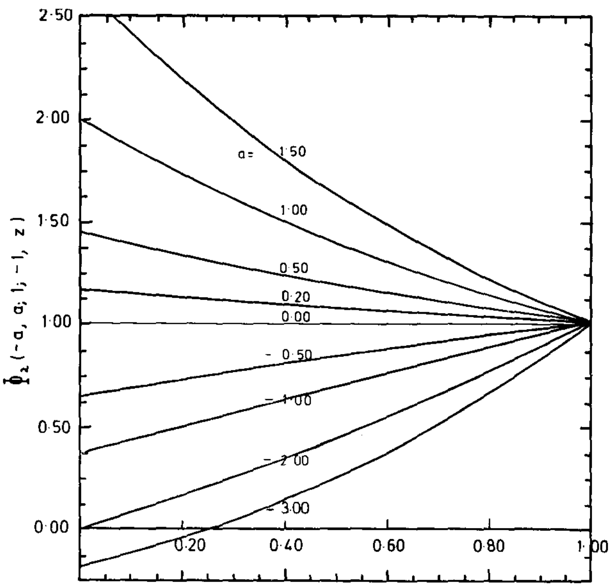

Fig. 3 The Hypergeometric Series Φ_2 $(-a,a,1;-1,z)$ as a Function of a and z

$$-\frac{B'_1(x)\,F(x,o)}{B_1(x)\,(s+k(x))}. \qquad (7)$$

In particular, if the functional forms $B_1(x)=x^\alpha$ and $k(x)=k_o x^\beta$ are used, the integral of equation (7) can be obtained and is given by

$$\bar{F}(x,s)=\frac{Cx^\alpha}{(s+k_o x^\beta)^{\alpha/\beta}} + \int_x^{x_m}\left(\frac{x}{x'}\right)^\alpha \frac{a}{x}\Big(F(x',o)-f(x',o)\Big)\frac{(s+k_o x'^\beta)^{\alpha/\beta-1}}{(s+k_o x^\beta)^{\alpha/\beta}}\,dx' \qquad (8)$$

The constant of integration, C, is obtained from the condition $F(x_m,t)=1$, and the Laplace tranform in equation (8) is then made up of standard forms that can be inverted direct[7] to give

$$F(x,t)=\left(\frac{x}{x_m}\right)^\alpha \Phi_2(-\alpha/\beta,\alpha/\beta;1;-k_o x_m^\beta t,-k_o x^\beta t) + \int_x^{x_m}\Big\{\Big(\frac{a}{x}F(x',o)-f(x',o)\Big)\left(\frac{x}{x'}\right)^\alpha e^{-k_o x'^\beta t}\,{}_1F_1(\alpha/\beta,1;z)\Big\}\,dx',$$

where $z=(x'^\beta-x^\beta)\,k_o t$. . . . . (9)

$\Phi_2(a,b;1;x,y)$ is the generalized hypergeometric series[8] in two variables, and ${}_1F_1(a,1;z)$ is the confluent hypergeometric function[9]. Unfortunately, no convenient tabulation of the generalised hypergeometric series is available, but it can easily be evaluated. The confluent hypergeometric function is well tabulated in reference (9). In general, analytical expressions are not available for the size-distribution functions of the initial batch, and the integral in equation (9) must consequently be evaluated by numerical quadrature. This poses no difficult computational problems.

SPECIAL CASES AND GENERALIZATIONS

Equation (9) includes all previous solutions as special cases. In particular, it takes on a simple form when the initial charge is all concentrated at a single size. In this situation, $f(x,o)=\delta(x-x_m)$, and the solution becomes

$$F(x,t)=\left(\frac{x}{x_m}\right)^\alpha \Phi_2(-\alpha/\beta,\ \alpha/\beta;\ 1;\ -k_o x_m^\beta t,\ -k_o x^\beta t) - \left(\frac{x}{x_m}\right)^\alpha e^{-k_o x_m^\beta t}\,{}_1F_1(a/\beta,\ 1-z) \text{ for } x<x_m;$$

with $z=(x_m^\beta-x^\beta)\,k_o t$ and $F(x_m,t)=1$. . (10)

In spite of the convenient form of this solution, the form chosen for the selection function is not versatile enough to describe the experimentally observed data. In particular, Kelsall and Reid[10], Loveday[11], and Austin[2]

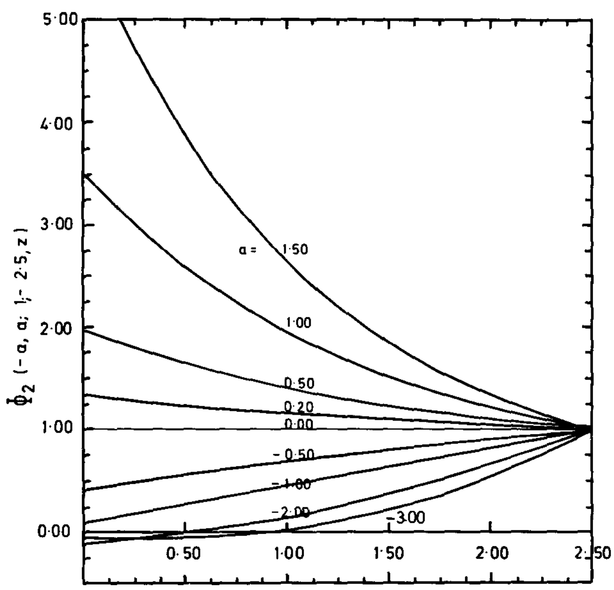

Fig. 4 The Hypergeometric Series Φ_2 $(-a,a,1;-2.5,z)$ as a Function of a and z

have shown that the selection function has a definite maximum and decreases at large sizes. So that the analytical solution can handle such data, it can be extended to include breakage functions of the type

$$k(x)=k_i\, x^{\beta_i} \quad x_{i-1} \le x \le x_i \qquad \ldots\ldots\ (11)$$

In other words, the selection function is described by a distinct power law in each of several size intervals. The size intervals are, of course, not meant to correspond to sieve mesh sizes. The solution is then a piece-wise smooth function as shown in the following

If $U_i = {}^{\alpha}/\beta_i - {}^{\alpha}/\beta_{i-1}$,

$\nu_i = -k_i x_{i-1}^{\beta_i} t$

and $\psi(x,x') = {}^{\alpha}/_x F(x',o) - f(x',o)$

$$F(x,t) = \left(\frac{x}{x_m}\right)^{\alpha} \Phi_2\Big(-{}^{\alpha}/\beta_m,\ U_m, \ldots\ldots\ldots, U_{m-j+1},$$

$${}^{\alpha}/\beta_{m-j},\ 1;\ -k_m x_m^{\beta_m} t,$$

$$\nu_m t, \ldots, \nu_{m-j}\, t,\ -k_{m-j} x^{\beta_{m-j}} t\Big)$$

$$+\sum_{i=1}^{j} \int_{x_{m-i}}^{x_{m-i+1}} \left(\frac{x}{x'}\right)^{\alpha} \psi\left(x_{m-i}, x'\right)$$

$$\Phi_2\Big(1-{}^{\alpha}/\beta_{m-i+1},\ U_{m-i+1}, \ldots\ldots\ U_{m-j+1},$$

$${}^{\alpha}/\beta_{m-j},\ 1;\ -k_{m-i+1}\, x'^{\beta_{m-i+1}} t,$$

$$\nu_{m-i+1} t, \ldots.\ \nu_{m-j+1}\, t,\ -k_{m-j} x^{\beta_{m-j}} t\Big)\, dx'$$

$$+\int_{x}^{x_{m-j}} \left(\frac{x}{x'}\right)^{\alpha} \psi(x,x')\ {}_1F_1\Big({}^{\alpha}/\beta_{m-j},\ 1,$$

$$k_{m-j}\, x' t - k_{m-j}\, x^{\beta_{m-j}} t\Big) \exp$$

$$\Big(-k_{m-j}\, x'^{\beta_{m-j}} t\Big)\, dx'$$

in the region $x_{m-j-1} \le x \le x_{m-j}$. . . . . (12)
For the largest size region, $j=o$.

NUMERICAL RESULTS

Two numerical solutions have been evaluated to demonstrate the general characteristics of the particle-size distributions. For each solution, a uniformly sized feed of particles of size x_m was assumed. In the first case, the selection function $k(x)=({}^{x}/x_m)^{0\cdot 8}$ and the breakage function $B(x,x')=({}^{x}/x')^{1\cdot 2}$ were used, and the particle-size distribution for various grinding times is drawn in Figure 8. In the second case, the same breakage function was used, but the selection function had a maximum and is given by

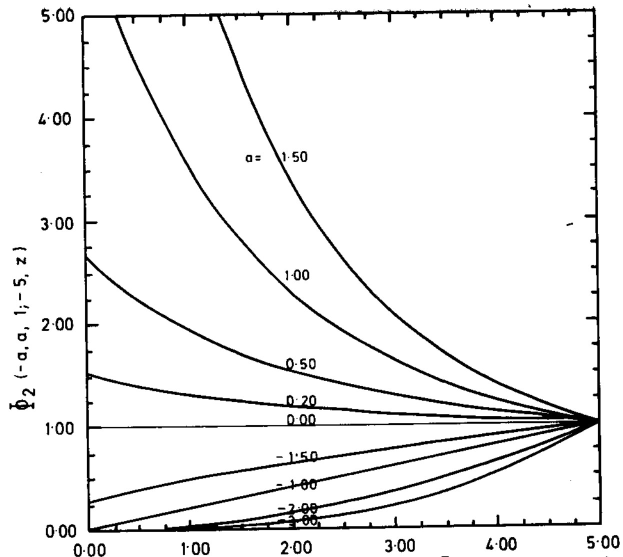

Fig. 5 The Hypergeometric Series Φ_2 (—a,a,1 ;—5,z) as a Function of a and z

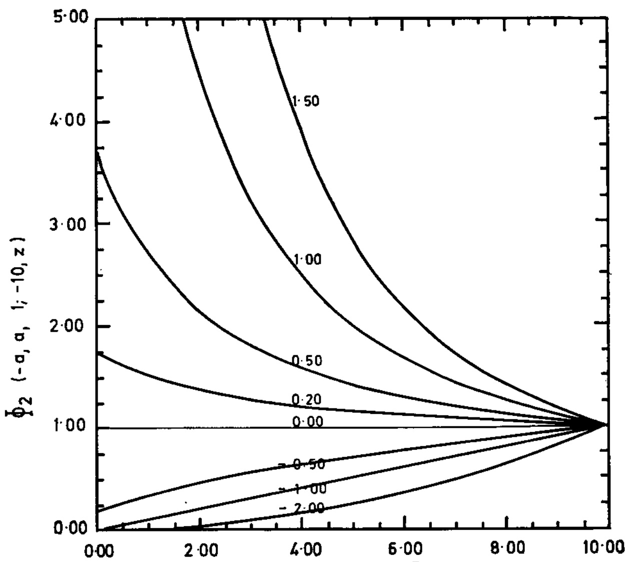

Fig. 6 The Hypergeometric Series Φ_2 (—a,a,1 ;10,z) as a Function of a and z

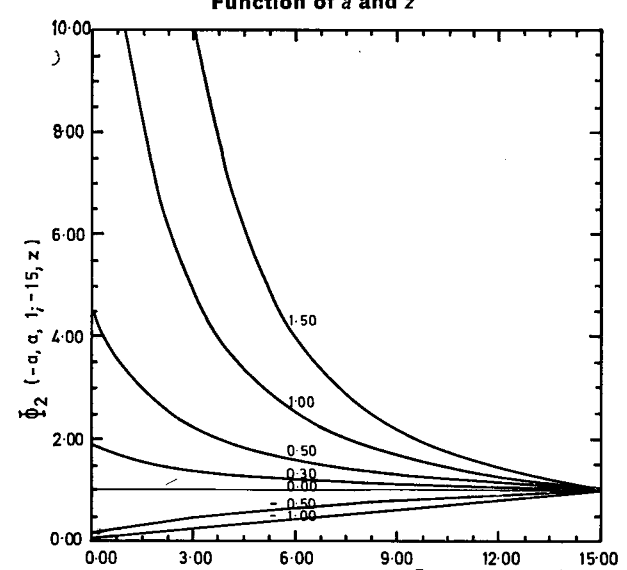

Fig. 7 The Hypergeometric Series Φ_2 (—a,a,1 ;15,z) as a Function of a and z

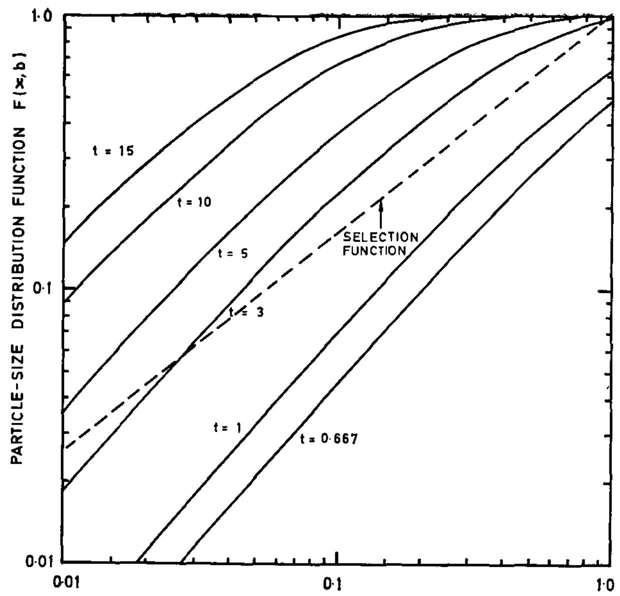

Fig. 8 Particle Size Distribution as a Function of Time

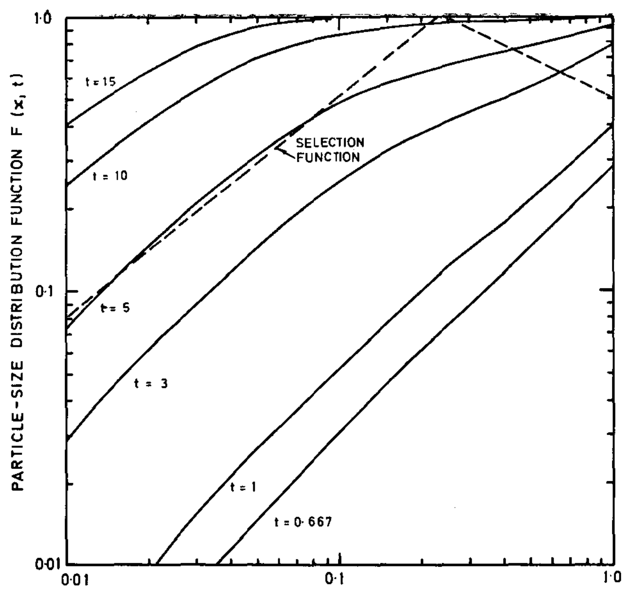

Fig. 9 Particle Size Distribution as a Function of Time

$$k\ (x)=0.5\left(\frac{x}{x_m}\right)^{-0.48} \text{for } 0.2365\ {}^{x}/x_m\leq 1$$

$$=0.315\ ({}^{x}/x_m)^{0.8} \text{ for } {}^{x}/x_m\leq 0.236.$$

The particle-size distribution for various grinding times is shown in Figure 9.

CONCLUSIONS

A convenient analytical solution to the batch-comminution equation has been obtained in terms of the hypergeometric functions. The solution for monotonic selection functions can be conveniently evaluated by use of the numerical values of the hypergeometric series that are presented graphically in the paper.

Selection functions that have a maximum are included in the analysis.

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