The effect of cost and price fluctuations on the optimum choice of mine cutoff grades

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SYNOPSIS

An analytical solution to the best choice of cutoff grade for a mine is deduced for an assumed model of selective mining. It is found that this choice is determined by the constancy of a special function, the Hamiltonian, over the life of the mine. It is suggested that the appropriate constant value for this function should be determined by the expected long-term average of the paylimit. In that case, the optimum profile can be calculated at each point in time according to the prevailing working cost and product price. In particular, a decision can be made between operation at the true paylimit or at the subsidized paylimit implied by either the lease and tax formulas or the assistance formula for gold mines. It is also shown how the basic formulation can be extended for use in the determination of the optimum joint paylimit for two products over the life of the mine.

SAMEVATTING

'n Analitiese oplossing wat betref die beste keuse van 'n afkapgraad vir 'n myn word afgelei vir 'n veronderstelde model van selektiewe mynbou. Daar is gevind dat die keuse bepaal word deur die bestendigheid van 'n spesiale funksie, die Hamilton-funksie, oor die lewensduur van die myn. Daar word aan die hand gedoen dat die gepaste konstante waarde vir hierdie funksie bepaal moet word deur die verwagte langtermyngemiddelde van die rendeergrens. In so 'n geval kan die optimale profiel op elke tydstip bereken word volgens die heersende bedryfskoste en produkprys. Daar kan veral besluit word of die myn op die ware rendeergrens, of op die gesubsidieerde rendeergrens deur òf die huur- en belastingformule vir goudmyne òf die bystandsformule vir goudmyne geïmpliseer, bedryf moet word. Daar word ook getoon dat die basiese formulering uitgebrei kan word vir gebruik by die bepaling van die optimale gesamentlike rendeergrens vir twee produkte oor die lewensduur van die myn.

Introduction

The cutoff grade used in the selection of mineral blocks is determined both by physical mining constraints and by legal requirements imposed by the State to encourage responsible exploitation of mineral resources. In particular, for South African gold mines, the lease stipulation requires that the average grade mined from reserves should be compatible with the prevailing paylimit[1,2].

Recent fluctuations in the price of gold have placed a considerable burden on mine planners attempting to define ore reserves, as well as exacerbating the position of marginal mines. In addition, consideration may have to be given to co-products, such as uranium, when a decision is taken on the viability of proposed stoping plans. It is therefore of interest to examine the economic consequences of optimum cutoff grades selected solely on the basis of price and cost fluctuations, and to analyse the difference between these and the paylimit or joint paylimit. In order to do this, it is necessary to formulate a cash flow model that describes both the selection mechanism for ore blocks and the statistical nature of the grade distribution in a mine.

Critical Assumptions

It is assumed that, although stoping is selective, development is not selective and the development grid will span the exploited lease area. It is therefore appropriate to express working costs as the sum of two components: the first related to stoping and treatment of ore, and the second dependent on the rate at which the development grid is extended to expose additional mining blocks. Thus, the total working cost per unit of ore mined increases as mining becomes more selective.

It is also assumed that the grade distribution of mineral blocks that are exposed by development remains uniform. This distribution would vary if, for example, the location of relatively rich areas of a mine were known and were exploited before the poorer areas.

Finally, the rather conservative assumption is made that, once mineral blocks have been rejected, they can never be reclaimed. The total production rate is assumed to be fixed, and is therefore sustained at each point in time by the appropriate combination of the rate at which mineral blocks are exposed by development and the fraction of these blocks selected for mining.

Cash Flow Model

The cash flow generated by a mine can be expressed for a single mineral product in terms of the following quantities:

F_0 = total size of the mineral deposit
L_0 = operating life of the mine at the given production capacity if the entire deposit is mined
$F(t)$ = rate at which mining blocks are exposed by development at time t
g = cutoff grade at time t
$Q(g)$ = fraction of the mineral blocks with grades equal to or above g
$\bar{g}(g)$ = average grade of the mineral blocks with grades equal to or above g
$\eta(\bar{g})$ = mining loss and efficiency of recovery
$C_d(t)$ = development cost per unit size of the deposit at time t
$c(t)$ = all mining and treatment costs per unit of ore mined at time t
$P(t)$ = price of the mining product per unit quantity recovered at time t.

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The units of these parameters depend on the nature of the deposit. For a tabular orebody, it is appropriate to assign units of area to F_0, area per year to $F(t)$, mineral mass per unit area to g and $\bar{g}(g)$, and cost per unit area to $C_d(t)$ and $c(t)$. An auxiliary quantity used to facilitate the subsequent analysis is the cumulative fraction of the deposit that has been exposed by development up to time t. This is denoted by H,

$$H=\frac{1}{F_0}\int_{t_0}^{t} F(t')dt' \; , \qquad \ldots\ldots\ldots\ldots (1)$$

where t_0 is the time at which production starts. If the time lag between development and stoping is assumed to be small, then, when $H=1$, the elapsed time t is equal to L, the total operating life of the mine. The instantaneous rate of production at time t is $Q(g)F(t)$. Since the mine production capacity is considered to be fixed, $Q(g)F(t)$ must be constant and is given by

$$Q(g)F(t)=F_0/L_0. \qquad \ldots\ldots\ldots\ldots (2)$$

From equations (1) and (2), time t can be expressed by the differential relationship

$$\frac{dt}{dH}=L_0Q(g) \; . \qquad \ldots\ldots\ldots\ldots (3)$$

The operation of the mine can thus be considered to occur over the fixed interval $0 \leqslant H \leqslant 1$, with the elapsed time $t(H)$ determined as a function of H by equation (3).

The cash flow rates for revenue and working costs are given respectively by

$$R(t)=\eta(\bar{g})\; P(t)\; \bar{g}(g)\; Q(g)\; F(t) \qquad \ldots\ldots\ldots (4)$$

and

$$W(t)=c(t)\; Q(g)\; F(t)+C_d(t)\; F(t) \qquad \ldots\ldots (5)$$

It is understood that all cost and price functions are expressed in deflated or 'purchasing power equivalent' terms. If, in addition, all discounting functions are included with the terms $P(t)$, $c(t)$, and $C_d(t)$, the net present worth of the cash flow generated by the mine is given by

$$I=\int_{t_0}^{L} Z(g,t)\; F(t)\; dt \; . \qquad \ldots\ldots\ldots\ldots (6)$$

where $Z(g,t)=\eta(\bar{g})\; P(t)\; \bar{g}(g)\; Q(g)-c(t)\; Q(g)-C_d(t) \qquad (7)$

It is required to find the variable cutoff grade $g(t)$ as a function of time, which will give the maximum value of I.

Optimum Cutoff Grade

Profiles of optimum cutoff grades have been determined in a number of cases by the numerical technique of 'dynamic programming'[3–7]. However, if the transformation $F_0dH=F(t)dt$ implied by equation (1) is used, the net present worth as given by equation (6) can be expressed as an integral over the fixed interval $0 \leqslant H \leqslant 1$. The determination of the maximum value of I, bearing in mind that t and g are linked by equation (3), is in fact equivalent to finding the solution to a basic problem in optimal control theory[8]. The solution is given in terms of a special function M defined by

$$M(t,g,H)=Z(t,g)+\lambda(H)L_0Q(g), \qquad \ldots\ldots\ldots (8)$$

where $\lambda(H)$ is an additional unknown function. Specifically, the best choice[9] of g as a function of H is such that M is constant, $\partial M/\partial g=0$, and $\lambda(1)=0$. If $q(x)dx$ is the fraction of mineral blocks with grades in the range x to $x+dx$, then

$$Q(g)=\int_g^{\infty} q(x)dx \text{ and } Q(g)\bar{g}(g)=\int_g^{\infty} xq(x)dx. \text{ Therefore,}$$

$$\frac{\partial}{\partial g}\Big[Q(g)\Big]=-q(g) \quad , \qquad \ldots\ldots\ldots\ldots (9)$$

$$\frac{\partial}{\partial g}\Big[\;Q(g)\bar{g}(g)\;\Big]=-g\;q(g)\;, \qquad \ldots\ldots\ldots (10)$$

$$Q\Big(g\Big)\frac{\partial}{\partial g}\Big[\;\bar{g}(g)\;\Big]=\Big(\;\bar{g}(g)-g)\;\Big)q(g) \; . \qquad \ldots (11)$$

By the use of these relationships and substitution of equation (7) into equation (8), it can be shown that

$$\frac{\partial M}{\partial g}=q\Big[P(\bar{g}-g)\bar{g}\frac{\partial \eta}{\partial \bar{g}}-\eta Pg+c-\lambda L_0\;\Big]$$

The requirement $\partial M/\partial g=0$ consequently implies that

$$P(t)[\bar{g}(g)-g]\bar{g}(g)\,\frac{\partial \eta(\bar{g})}{\partial \bar{g}}-\eta(\bar{g})P(t)g+c(t)-$$
$$\lambda(H)L_0=0 \; . \qquad \ldots\ldots\ldots\ldots\ldots (12)$$

By elimination of $\lambda(H)$ between equation (12) and equation (8), and the employment of definition (7) of Z, it follows that

$$M=P(t)Q(g)\Big[\bar{g}(g)-g\;\Big]\frac{\partial}{\partial \bar{g}}\Big[\bar{g}(g)\eta(\bar{g})\;\Big]-C_d(t) \quad (13)$$

Equation (13) defines the optimal grade profile as an implicit function of g provided the appropriate constant value of M has been found. A method for finding M is discussed in the next section. Function M is often referred to as the Hamiltonian owing to the close connection between the structure of the present problem and W. R. Hamilton's formulation of the equations of motion in classical mechanics. The constancy of the Hamiltonian is analogous to the principle of energy conservation and, in the present case, arises because the function $Z(g,t)$ given by equation (7) does not depend explicitly[10] on the independent variable H.

Profiles of Optimal Cutoff Grade

The value of the Hamiltonian M can be found if the optimum cutoff grade is known at one point during the life of the mine. At the end of the mine life, $H=1$ and $t=L$. By use of the boundary condition $\lambda(H=1)=0$, equation (12) defines g in terms of $P(L)$ and $c(L)$ but, unfortunately, L is not known. However, L can be found by the following trial-and-error procedure.

(a) Assume a particular value for the end of the mine life, L.
(b) Compute the value of g at $t=L$ or $H=1$ by solving equation (12) with $\lambda=0$ and the assumed values of $P(L)$ and $c(L)$.
(c) Substitute $g(L)$, $P(L)$, and $C_d(L)$ into equation (13) to find the value of M.
(d) Using this value of M, solve equation (13) for g at sufficient points in time between t_0 and L to define $g(t)$.

(e) Starting at $t=t_0$, integrate the elapsed time relationship (3) to determine the value of L_0 that corresponds to the value of L assumed in step (a). Explicitly, L_0 is given by

$$L_0 = \int_{t_0}^{L} [1/Q(g(t))]dt \quad . \quad . \quad . \quad . \quad (14)$$

(f) Repeat steps (a) to (e) until L_0 corresponds to the value appropriate to the size of the deposit and the production capacity of the mine.

As an example, consider a hypothetical mine with a grade distribution described by an exponential function of the form

$$q(x) = \frac{1}{\gamma} e^{-x/\gamma} \quad . \quad . \quad . \quad . \quad (15)$$

where γ is the average grade of the entire deposit (i.e. all the mineral blocks above and below the paylimit).

It can be shown by integration that $Q(g)=\exp(-g/\gamma)$ and $\bar{g}-g=\gamma$. If the efficiency term $\partial[\bar{g}\eta]/\partial\bar{g}$ is constant, an effective product price, $p(t)$, can be defined as

$$p(t) = P(t)\frac{\partial}{\partial \bar{g}}\left[\bar{g}(g)\,\eta(\bar{g})\right] = \bar{\eta}P(t) \quad . \quad . \quad . \quad . \quad (16)$$

If no discount factors are applied, and if the deflated development cost $C_d(t)$ remains constant, equations (12) and (13) can be shown to define the optimal cutoff grade as

$$g(t) = \gamma \ln[p(t)/p(L)] + c(L)/p(L). \quad . \quad . \quad . \quad . \quad (17)$$

Using an asterisk to denote actual undeflated quantities, let $c^*(t)$ be the unit working cost not related to development and $p^*(t)$ the gold price received at time t. Then, if the unit working cost, $c^*(t)$, is used as an inflation index, the deflated price received by the hypothetical mine is

$$p(t) = p^*(t)[c^*(b)/c^*(t)], \quad . \quad . \quad . \quad . \quad (18)$$

where $t=b$ is an arbitrary base time. Hence,

$$p(t)/p(L) = [p^*(t)/p^*(L)]\,[c^*(L)/c^*(t)] \quad . \quad . \quad . \quad . \quad (19)$$

and

$$c(t)/p(t) = c^*(t)/p^*(t). \quad . \quad . \quad . \quad . \quad (20)$$

Suppose that $c^*(t)$ and $p^*(t)$ correspond to the average values of the quarterly results for gold producers published by the Chamber of Mines of South Africa. The values of $p(t)/p(L)$ and $c(t)/p(t)$ computed from equations (19) and (20) using the quarterly results from 1971 to mid 1982 are plotted in Fig. 1. In this diagram, time L is taken to be at the end of the second quarter in 1982.

The points of optimal cutoff grade, computed from equation (17) are plotted as circles in Fig. 1. It is immediately obvious that the optimal points are in direct contrast to the paylimit values (plus signs) computed from equation (20). The optimal points follow the relative changes in gold price in such a way as to increase the production of gold when the price is high and to reduce the production of gold during periods of low prices. This qualitative observation has been made previously for products other than gold[11,12].

Effect of Cost and Price Fluctuations

If costs and prices fluctuate, the possibility arises that there would be several profiles of optimum cutoff grade, each of which would satisfy the reserve constraint represented by equation (14). For example, if the price is assumed to oscillate sinusoidally and if the deflated cost is assumed to be constant, the paylimit would

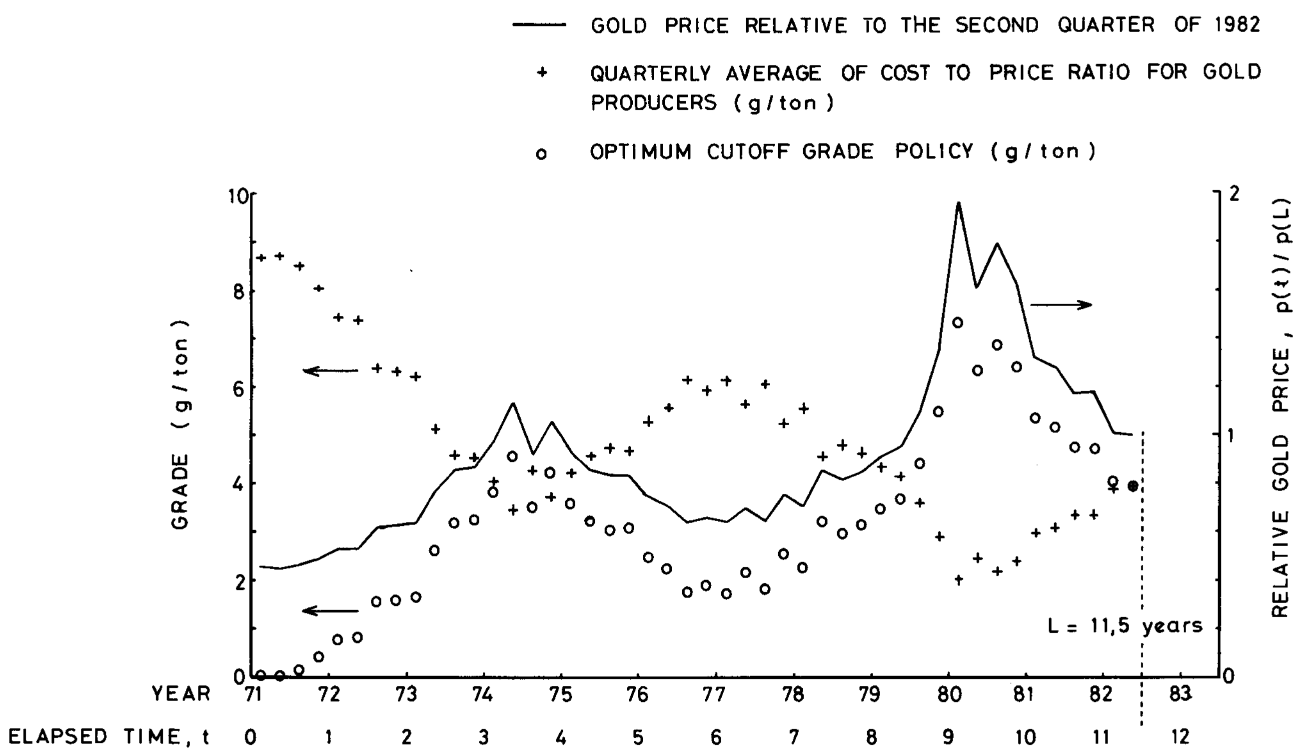

Fig. 1—Optimum cutoff grade for a paylimit corresponding to quarterly averages of costs in the gold-mining industry and of prices since 1971.

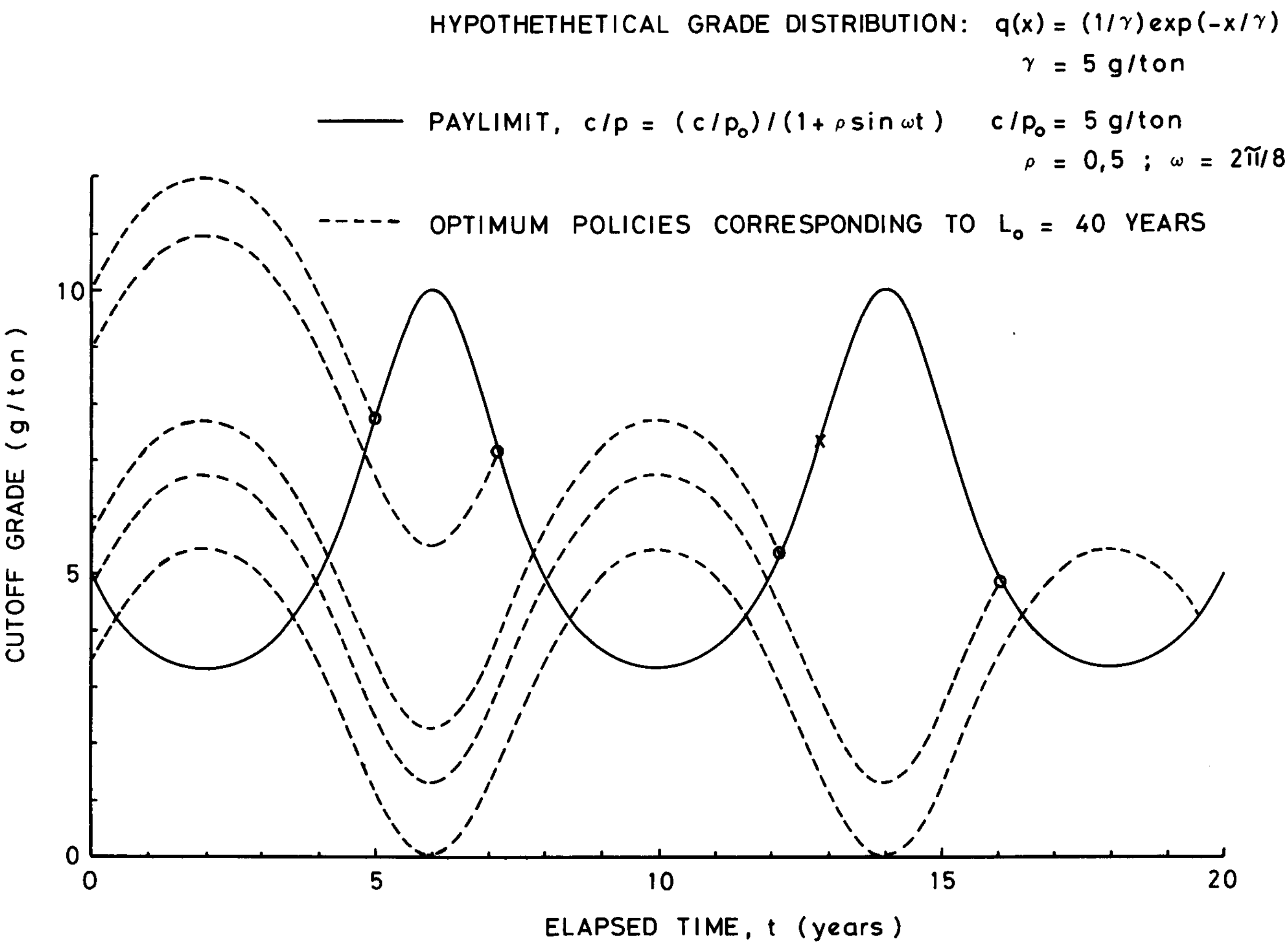

Fig. 2—A family of profiles showing locally optimum cutoff grades for the reserve constraint, $L_0=40$ years. The profiles depict mining policies of decreasing average grades over increasing lengths of time.

fluctuate as shown by the solid line in Fig. 2. The family of profiles of optimum cutoff grade, represented by the broken lines intersecting the paylimit curve at different values of L, all satisfy the constraint $L_0=40$ years. It must be decided which of these L values will yield the maximum net present worth, I. This can be determined by direct calculation of I for each case. However, from the considerations presented below, it appears that the best profile can be estimated from a knowledge of the average cost and price levels only and that the operating life, L, does not have to be predetermined. By the substitution of equation (8) into equation (6) it can be shown that I is given by

$$I = F_0\left(M + \frac{1}{L_0}\int_{t_0}^{L}(pg-c)dt \right). \quad . . . (21)$$

If the deflated working cost and development cost, c and C_d, are constant and if the grade distribution is exponential, then from equations (13) and (16),

$$M/c+C_d/c=\exp(-c/p(L)\gamma)/(c/p(L)\gamma). \quad . . . . . (22)$$

and

$$I/F_0c + C_d/c = (M/c + C_d/c) + \frac{1}{L_0}\int_{t_0}^{L}(pg/c-1)dt. \quad . . . . (23)$$

Equation (17) gives g as a function of time. By the use of specific parameter values of $p=p_0(1+\rho\sin\,\omega t)$ ($\rho=0{,}5$, $\omega=2\pi/8$, and $c/p_0=1$), equations (22) and (23) can be evaluated for each value of L shown in Fig. 2. The results are given in Table I.

TABLE I

OPTIMUM RETURNS FOR DIFFERENT MINE LIVES SATISFYING THE SAME RESERVE CONSTRAINT, $L_0=40$ YEARS

Operating life L year	Final cutoff grade c/p_L g/t	Scaled Hamiltonian $(M+C_d)/c$	Scaled total return $I/F_0c+C_d/c$
5,01	7,76	0,136	0,364
7,18	7,16	0,167	0,346
12,17	5,35	0,320	0,464
16,06	4,88	0,386	0,409
19,54	4,25	0,503	0,449
Paylimit: 12,87	7,32		0,350

The last line in Table I represents the return obtained when the cutoff grade is equal to the paylimit, c/p. It can be seen that the best of the local optima occurs at $L=12{,}17$ years and is more than 30 per cent greater than the paylimit policy. The best choice of L, in fact, appears to be controlled by the average paylimit. In the case where the effective product price is given by equation (16), the Hamiltonian (equation (13)) becomes

$$M = pQ(\bar{g} - g) - C_{\mathrm{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

From equations (9) to (11), it follows that

$$\partial g/\partial M = -1/pQ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Furthermore, since equation (24) defines g as an implicit function of M, L is in turn an implicit function of M when the fixed reserve constraint L_0=constant is applied. It can be shown by the differentiation of equation (14) that

$$\partial L/\partial M = (Q \,\big|\, t{=}L) \int_{t_0}^{L} (q/pQ^3)\, dt > 0 \quad . \quad . \quad . \quad . \quad (26)$$

The expected total return, I, can be evaluated approximately if representative averages, c_{av} and p_{av}, can be assigned to the deflated cost and price levels. From equation (21), the expected value of the return is thus

$$\hat{I} = F_0 \left(M + \frac{p_{\mathrm{av}}}{L_0} \int_{t_0}^{L} g dt - \frac{c_{\mathrm{av}}}{L_0} (L - t_0) \right) \quad . (27)$$

The best choice of L to maximize $\hat{I}$ should therefore occur when $\partial \hat{I}/\partial M = 0$. By differentiation of equation (27),

$$\frac{\partial \hat{I}}{\partial M} = F_0 \Big(1 + \frac{p_{\mathrm{av}}}{L_0} \int_{t_0}^{L} \frac{\partial g}{\partial M} dt + \frac{p_{\mathrm{av}}}{L_0} [g \,\big|\, t{=}L] \frac{\partial L}{\partial M} - \frac{c_{\mathrm{av}}}{L_0} \frac{\partial L}{\partial M} \Big) \; .$$

From equations (25) and (14),

$$\int_{t_0}^{L} \frac{\partial g}{\partial M} dt \approx -L_0/p_{\mathrm{av}} \; .$$

Consequently, $\partial \hat{I}/\partial M = 0$ if $g(L) = c_{\mathrm{av}}/p_{\mathrm{av}}$ . . . . (28)

Hence the optimum choice of L occurs for the final grade c/p_L that is as close as possible to the average paylimit $c_{\mathrm{av}}/p_{\mathrm{av}}$. This is confirmed in Table I, where the final grade that corresponds to the best return is closest to the average paylimit of 5,77 g/t for the assumed parameters.

The most important consequence of equation (28) is that, provided an estimate of the average deflated cost and price parameters, c_{av} and p_{av}, is available, a good estimate of the value of the Hamiltonian can be made. Based on this value, the optimum cutoff grade can be generated at any point in time without predetermination of the optimum operating life, L.

Grade Constraints and Tax and Lease Payments

A number of factors prevent the direct implementation of the cutoff grade policies illustrated above. In particular the mining profit may become unacceptably small or even negative during the suggested periods of mineral conservation when the product price is low. Conversely, legal constraints may curtail overmining during periods when the product price is high. These restrictions can, in general, be incorporated in the formulation of the problem by the addition of extra terms in the definition of the Hamiltonian function, M.

Apart from these restrictions, the effects of tax and lease payments have not been included in the basic formulation of the cash flow rate for the mine. If the initial tax-free period of a gold mine is ignored, the net cash flow rate after tax is obtained by a slight modification of equation (7):

$$Z(g,t)F(t) = \alpha R(t) - \beta W(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where α and β are constants such that $1 > \alpha > \beta > 0$. (See Krige[16].) The inclusion of tax and lease payments effectively lowers[16] the paylimit by the ratio β/α. The lowering of the paylimit is an implicit subsidy by the State that encourages greater extraction of the mineral deposit. In the case of a mine qualifying for State assistance, the constants α and β are altered to increase this subsidy and enable the mine to extend its life and avoid premature closure. This additional reduction in the paylimit during unfavourable economic conditions is, in fact, similar to the conservation behaviour of the optimum profiles shown in Figs. 1 and 2.

The basic question as to whether a mine should operate at the paylimit or at the subsidized paylimit or should apply for State assistance can be addressed to some extent by the use of the economic model formulated above. If, for example, a mine has discretion in operating between the paylimit and the subsidized paylimit, the cutoff grade is bounded at each point in time, t, by the two constraints

$$c(t) - p(t)g \geqslant 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and

$$\alpha p(t)g - \beta c(t) \geqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

These constraints can be incorporated in the model by the definition of a modified Hamiltonian:

$$M_{\mathrm{c}} = \alpha p(t) Q(g)\bar{g}(g) - \beta c(t) Q(g) - \beta C_{\mathrm{d}}(t) + \lambda(H) L_0 Q(g) + \mu_1(H)[c(t) - p(t)g] + \mu_2(H)[\alpha p(t)g - \beta c(t)]. \quad . (32)$$

The optimum grade profile is such that M_{c} is constant and

$$\partial M_{\mathrm{c}}/\partial g = -q[\alpha p(t)g - \beta c(t) + \lambda(H)L_0] - \mu_1(H)p(t) + \mu_2(H)\alpha p(t) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The additional functions that were introduced, $\mu_1(H)$ and $\mu_2(H)$, must be chosen to be positive or zero in such a way that

(a) if $c(t) - p(t)g = 0$, $\mu_1 \geqslant 0$, $\mu_2 = 0$,
(b) if $c(t)/p(t) > g > \beta c(t)/\alpha p(t)$, $\mu_1 = 0$, $\mu_2 = 0$,
(c) if $\alpha p(t) - \beta c(t) = 0$, $\mu_1 = 0$, $\mu_2 \geqslant 0$.

The condition $\partial Mc/\partial g = 0$ must be evaluated and tested for compatibility against (a), (b), and (c) above to determine whether the cutoff grade falls between or on the upper or lower bound. For further details of this technique, reference can be made to Bryson and Ho[8], Chapter 3.

As an example, it is assumed that the cutoff grade is required to fall between the average quarterly cost-to-price ratios of producing gold mines, depicted by plus signs in Fig. 3, and a subsidized paylimit that is 16 per cent lower ($\beta/\alpha = 0{,}84$). The subsidized paylimit is depicted by the crosses in Fig. 3, and the optimum profile falling on or between the two constraints is

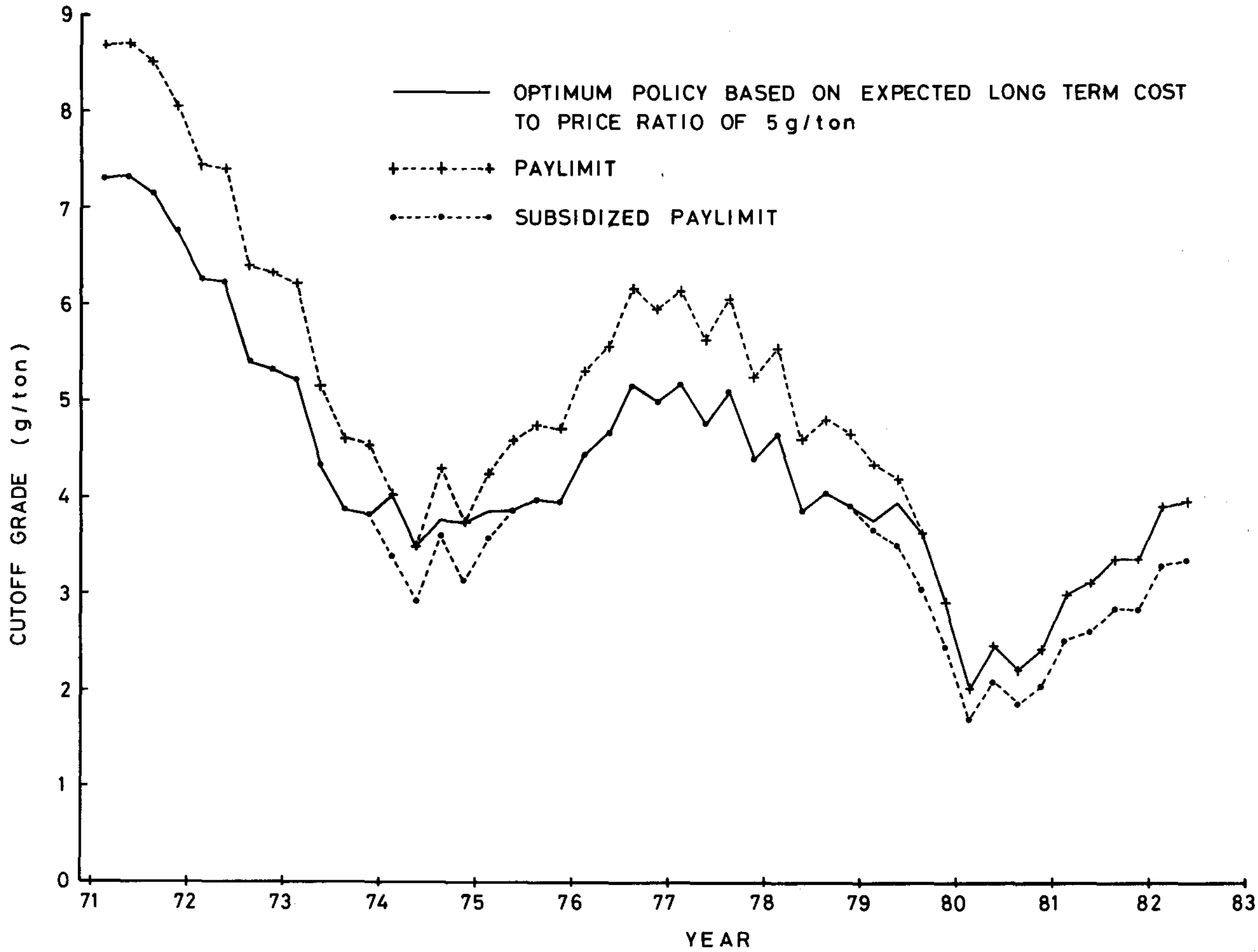

Fig. 3—Constrained optimum cutoff grade.

shown as a continuous line. The optimum cutoff grade is computed from a Hamiltonian evaluated at an expected long-term paylimit of 5 g/t. For numerical convenience, the grade distribution is taken to be the same as in the previous examples.

If the deflated working cost is constant, the total return can be expressed in the scaled form

$$I/F_0c+C_d/c = (1/L_0)\left(\int_{t_0}^{L}(\bar{g}/g_{p1})dt-(L-t_0)\right), \quad (34)$$

where $g_{p1}=c/p(t)$ is the paylimit and L_0 is given by equation (14). The value of L_0 that corresponds to operating at the subsidized paylimit in Fig. 3 is $L_0=27{,}4$ years. If this value is applied to the actual paylimit and the profiles of optimum cutoff grade in Fig. 3, the operating lives, L, are as shown in Table II. The scaled return (34) for these lives is also shown in the table, together with the return corresponding to the optimum unconstrained profile given in Fig. 1. In the case of the unconstrained profile, $L_0=23{,}1$ years, a smaller deposit area than the assumed maximum of $L_0=27{,}4$ years. The unconstrained return is nevertheless more than 20 per cent greater than the subsidized paylimit policy.

The most interesting observation is that the subsidized paylimit yields a very much better return than the actual paylimit since the operating life is prolonged from 8,6 to 11,5 years, thereby gaining the benefit of the higher product price. The optimum constrained profile is only slightly better than the subsidized paylimit but is much smoother (Fig. 3) and is computed on the rationale of an expected stability in the long-term cost and price cycles.

TABLE II

COMPARISONS OF RETURNS FOR DIFFERENT CUTOFF POLICIES APPLIED TO HISTORICAL COST AND PRICE DATA

Cutoff grade policy	Operating life at zero cutoff L_0 years	Actual operating life L years	Scaled return before tax $I/F_0c+C_d/c$
Paylimit	27,4	8,6	0,301
Subsidized paylimit	27,4	11,5	0,417
Optimum constrained	27,4	11,1	0,424
Optimum unconstrained	23,1	11,5	0,515

Efficiency of Recovery

The overall recovery, η, depends both on the mining losses, reflected by the mine call factor, and on the efficiency of recovery. If the mining loss component reduces the average grade mined, g, by a constant factor, m, the head grade is

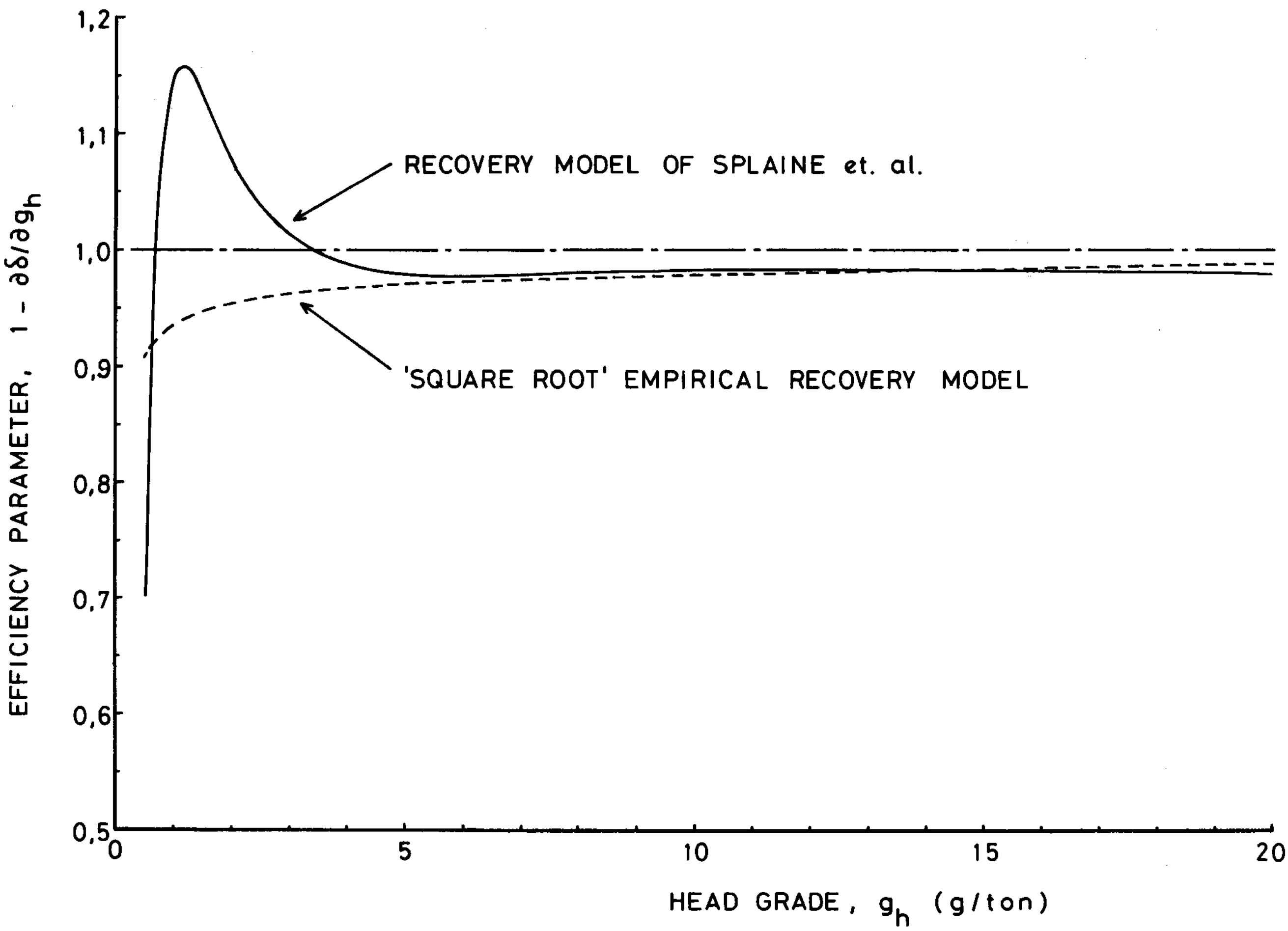

Fig. 4—Influence of head grade on the effective recovery efficiency.

$$g_h = m\bar{g}. \qquad (35)$$

After Splaine *et al.*[15], the efficiency of recovery can be defined as $1-\delta/g_h$, where δ is the final residue grade. The overall recovery is therefore

$$\eta = m(1-\delta/g_h) = m - \delta/\bar{g}\ . \qquad (36)$$

Since $\partial\delta/\partial g = m\ \ \partial\delta/\partial g_h$, the efficiency term $\partial[\bar{g}\eta]/\partial\bar{g}$ appearing in equation (13) can be reduced to

$$\bar{\eta} = \partial[\bar{g}\eta]/\partial\bar{g}. = \underline{m}(1-\partial\delta g_h) \qquad (37)$$

By use of the model for gold-plant recovery proposed by Splaine *et al.*[15] for δ, it can be shown that

$$1-\partial\delta/\partial g_h = (1-\exp(-\beta T))\exp(-L/g_h)\ (1+Lg_h)$$
$$(1+L^2 2g_h{}^2) + \epsilon\exp(-\beta T)\exp(-A/g_h)(1+A/g_h)(1$$
$$+A^2/2g_h{}^2) \qquad (38)$$

This model is compared[15] with an empirical 'square root' model for which

$$1-\partial\delta/\partial g_h = 1-k/2\sqrt{g_h}. \qquad (39)$$

With the numerical values used by Splaine *et al.*[15] for βT, ϵ, L, A, and k, equations (38) and (39) can be plotted as shown in Fig. 4. It is apparent that, provided the head grade is not less than 4 g/t, both models yield nearly constant and equivalent values of $1-\partial\delta/\partial g_h$. In this operating range, an overall constant efficiency, $\bar{\eta}$, can be used as in equation (16). The value of $\bar{\eta}$ is controlled mainly by the factor for mining loss, m.

The effect of $\bar{\eta}$ on the cutoff grade can be inferred from equation (12) since the end-point condition $\lambda(1)=0$ implies that

$$g_L = c(L)/\bar{\eta}P(L). \qquad (40)$$

The final grade, g_L, is therefore greater than the pay-limit $c(L)/P(L)$ for $\bar{\eta}<1$. As $\bar{\eta}$ decreases, the values of optimum cutoff grade are increased.

Optimum Joint Cutoff Grade

In this section a brief outline is given of how the model can be extended to solve for the optimum joint cutoff grade as a function of time. Suppose that the deposit contains two minerals that are denoted as g and u. The fraction of the deposit with the grade of mineral g between x and $x+dx$ and the grade of mineral u between y and $y+dy$ is given by the joint distribution[13,17] $f_{gu}(x,y)dxdy$. Let the mining blocks be divided into a set of n contiguous, non-overlapping grade categories with respect to the mineral u. Let S_i be the fraction of the deposit with u grades in the range $y_i - \triangle_i/2$ to $y_i + \triangle_i/2$. S_i is given approximately by

$$S_i \approx \triangle_i \int_0^\infty f_{gu}(x,y_i)dx. \qquad (41)$$

Let $Q_i(g_i)$ be the fraction of S_i for which mineral g grades are not less than g_i. Then

$$Q_1(g_1)S_1 \approx \triangle_1 \int_{g_1}^{\infty} f_{gu}(x,y_1)dx. \qquad (42)$$

Let $\bar{g}_1(g_1)$ be the average grade of mineral g within the fraction $Q_1(g_1)S_1$. Then

$$\bar{g}_1(g_1)\,Q_1(g_1)S_1 \approx \triangle_1 \int_{g_1}^{\infty} x f_{gu}(x,y_1)dx. \qquad (43)$$

The total fraction of developed blocks that are selected for mining when a set of g_1 values is specified is given by

$$Q = \sum_{i=1}^{n} Q_1(g_1)S_1. \qquad (44)$$

If $F(t)$ is the rate at which new blocks are exposed by development at time t, the rate of revenue generation is given by

$$R(t) = \sum_{i=1}^{n} [p_g(t)\bar{g}_1(g_1) + p_u(t)y_1]\,Q_1(g_1)S_1F(t), \qquad (45)$$

where $p_g(t)$ and $p_u(t)$ are the effective prices of the g and u products respectively. When the transformation $F(t)dt = F_o dH$ obtained from equation (1) is applied, the total return of the generated cash flow is given by

$$I = F_o \int_0^1 \left(\sum_{i=1}^{n} [p_g(t)\bar{g}_1(g_1) + p_u(t)y_1 - c(t)]\,Q_1(g_1)S_1 - C_d(t) \right) dH. \qquad (46)$$

Since the mine production capacity is assumed to be fixed (equation (2)), the elapsed time must obey the relationship

$$dt/dH = L_o \sum_{i=1}^{n} Q_1(g_1)S_1. \qquad (47)$$

The particular set of functions $g_1 = g_1^*(H)$ that yield the maximum value of I while obeying constraint (47) will be such that the Hamiltonian, M, defined below, is constant and that $\partial M/\partial g_1 = 0$.

$$M = \sum_{i=1}^{n} [p_g(t)\bar{g}_1(g_1) + p_u(t)y_1 - c(t) + \lambda(H)L_o]\,Q_1(g_1)S_1 - C_d(t). \qquad (48)$$

This function is directly analogous to equation (8). $\lambda(H)$ is to be found as part of the solution and must satisfy the boundary condition $\lambda(1) = 0$. From equations (42), (43), and (48) it can be shown that

$$\partial M/\partial g_1 = -\triangle_1 f_{gu}(g_1,y_1)[p_g(t)g_1 + p_u(t)y_1 - c(t) + \lambda(H)L_o]. \qquad (49)$$

Hence, the condition $\partial M/\partial g_1 = 0$ becomes

$$g_1 = [c(t) - p_u(t)y_1 - L_o\lambda(H)]/p_g(t), \qquad (50)$$

which defines the relationship for joint cutoff grade at any time during the life of the mine. At the end of the mine life, $\lambda(H=1) = 0$ and the joint cutoff grade is

$$g_1 = [c(t(1)) - p_u(t(1))y_1]/p_g(t(1)). \qquad (51)$$

If $p_u(t)$, y_1, or $\lambda(H)$ become sufficiently large, g_1 may become negative and thus meaningless in a physical sense. However, it must be noted that the joint distribution $f_{gu}(x,y)$ can be continued mathematically as $f_{gu}(x,y) = 0$ for all negative values of x or y. Consequently, the setting of g_1 to a negative value in equation (48) is equivalent to the setting of g_1 to zero.

The solution of the joint cutoff grade requires that g_1 be evaluated from equation (51) at an assumed value of $t(1) = L$. The appropriate value of M is then obtained by substitution of these values of g_1 and $\lambda(1) = 0$ into equation (48). At values of $t < L$, the g values are related to λ by equation (50). Hence, the right-hand side of equation (48) becomes a function of λ only, which can be solved to yield the required value of M.

Summary

The optimum cutoff grade policy for a particular model of selective mining has been demonstrated to correspond to the solution to a basic problem in the theory of optimal control. The economic model assumes that the mine operates at a constant milling rate, that no reclaiming of rejected mineral blocks can occur, and that the grade distribution from which selection is made does not change with time. The solution is such that the mineral product is conserved during periods when the price is low and is exploited when the price is high. The application of the model must be based either on postulated trends in costs and prices or, less restrictively, on the expected long-term average of cost and price levels.

The examples considered show that the optimum cutoff policy can generate a much larger return than the return obtained by the setting of the cutoff grade equal to the paylimit. In each case, a similar fraction of the mineral deposit is extracted. The optimum policy can therefore increase the overall income both to the State and the mine, although the accompanying fluctuations in cash flow may have to be smoothed by suitable policies of income retention. The solution technique can be modified if cutoff grades are constrained to fall between limits, such as the actual paylimit and the subsidized paylimit, determined by tax and lease considerations. When operation is between these limits, the profile of optimum cutoff grade is found to have the desirable property of being smoother, and therefore easier to follow, than short-term fluctuations in the paylimit. In addition, the impact of State assistance on mine profitability can be assessed if the subsidized paylimit is set according to the assistance formula.

The basic model includes the possibility of a functional relationship between the recovery efficiency of the product and the average grade of ore mined. However, based on a recently proposed model of plant recovery, it is found that the recovery efficiency can be represented by a constant value provided the head grades are above a critical threshold. Finally, the model can be extended for use in the determination of the optimum joint cutoff grade for two products.

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Survival in the metals industries

The recent recession has brought major changes in the structures of the copper, lead, zinc, aluminium, and nickel industries, according to a major study just completed by Commodities Research Unit. It concludes that many companies, if they are to prosper or even survive, will have to rethink traditional corporate policies to meet the changed conditions in the metals industries.

The study notes that, although prices for the five metals are likely to improve this year, this will not necessarily mean a return to profitability for all producers. At least three of the five metals — copper, lead, and nickel — are expected to have excess production capacity for the next five years, which will continue to put pressure on profit margins. However, the outlook for aluminium and zinc is more promising.

In regard to prospects for the next five years, the study sees fundamental changes in the supply–demand picture, with increased polarization between the large consuming areas and exporters in Developing Countries. The concentration of production in developing areas, dependent on exports and less concerned with making a commercial profit, is expected to result in a much slower response to price fluctuations than in the past. As a result, the study suggests companies in the metal industries should re-examine some of their cherished basic policy concepts. It suggests, for example, that producer prices should be abandoned in favour of free market rates, not only for metal sales and purchases, but also for concentrates and power. The need for long-term supply contracts is also questioned.

The study recommends that companies should also re-examine their financing and investment strategies in view of the need to become more flexible. It argues that high priority should be given to the reduction of corporate debt and that investment in high cost plant, equipment, and exploration projects should be critically reviewed.

Further details on the study are available from

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