

# The location of critical slip surfaces in slope-stability problems

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## SYNOPSIS

This paper illustrates the application of numerical modelling methods, in conjunction with limit-equilibrium techniques, to the location of critical slip surfaces in slope-stability problems.

Initially, the Spencer slice-based method of analysis is introduced, together with a dynamic-programming algorithm for slip-surface location. This limit-equilibrium method is assessed critically through the use of an illustrative problem.

The technique is then extended to locate the critical slip surface for a similar slope, but with more complex constitutive and yield relationships for the material comprising the slope. To this end, the use of the stress field generated from a finite-difference program, together with a dynamic-programming search routine, is presented as an advanced technique of slope-stability analysis for engineers.

## SAMEVATTING

Hierdie artikel illustreer die toepassing van getalprogrammeringsmetodes tesame met grensewewigtegnieke vir die ontleding van die kritiese swigtingsvlakligging in hellingstabiliteitsprobleme.

Aanvanklik is die Spencer skyfgebaseerde metode van analise bekend gestel, met die dinamiese programmeringsalgoritme vir die swigtingsvlakaanwysing. Hierdie grensewewigmetode word dan krities beoordeel deur die gebruik van 'n verduidelikende probleem.

Die tegniek word dan uitgebrei om die kritiese swigtingsvlak van 'n gelyksoortige helling aan te dui, deur gebruik te maak van meer ingewikkelde materiaalsamestellings en swigtingsverhoudinge. Die gebruik van die spanningsveld voortgebring deur die gebruik van 'n grens onderskeidsprogram tesame met 'n dinamiese programmeringsoekroetine is aangebied as 'n gevorderde hellingstabiliteit-ontledingstegniek vir ingenieurs.

## INTRODUCTION

Engineers generally use slope-stability analysis to identify the most probable failure geometry from a consideration of the shape of the critical slip surface. They also make use of many different analytical techniques to assess the stability of man-made or natural slopes. Predominant among these methods are the limit-equilibrium techniques, in which analysis by means of empirically developed field-stability charts may in certain circumstances be applicable. On the whole, however, more precise solutions based on site exploration and laboratory testing are generally recommended. The results of such an analysis are usually reported as 'factors of safety' (FOS) (i.e. the ratio of available strength to mobilized strength along the failure surface), or as a probability of failure for the particular geometrical and geotechnical parameters.

The limit-equilibrium technique, although widely used, is deficient in certain respects: primarily, the inability of the method to account for the more complex constitutive relationships of the materials comprising the slope. In addition, the effects of initial stresses are not easily incorporated, assumptions are made by which pore-pressure effects are generalized, and it is not possible to calculate local factors of safety. However, the location of the critical slip surface is found relatively easily and efficiently if the attendant simplification of the problem is acceptable or commensurate with the extent of the geotechnical data available.

Recent developments in the solution of boundary-value problems involving sets of differential equations have been applied, in the form of finite element (FE) or finite difference (FD) methods, to the analysis of slope stress and deformation. With the FE/FD techniques, a more representative multi-layer model can be analysed in conjunction with more complex constitutive and yield relationships than the simplistic (but often adequate) elasticity-based Mohr–Coulomb limit-equilibrium approach allows. Although FE and FD methods can more accurately predict the overall stress-deformation behaviour of a slope, unless a revised methodology of stability analysis is introduced or a stability criterion other than that of the FOS is used, the absence of an overall FOS renders the method unsuitable for the requirements of a classical stability analysis.

Through a combination of the above analytical techniques, a useful technique of stability assessment can be derived. With limit-equilibrium techniques, since the position of the critical failure surface is not known *a priori*, a numerical or multistage optimization process is used to determine its position. Also, recent research has highlighted the use of multistage optimization when applied to the analysis of the elemental stresses calculated from an FE method to resolve the most critical location of the slip surface.

This paper illustrates the application of critical slip-surface location techniques based on the numerical programming method to limit-equilibrium problems. The technique is then extended to locate the critical slip surface for a similar slope, but by the use of more complex constitutive and yield relationships, through the analysis of the finite-difference stress field generated from the FLAC (Fast Lagrangian Analysis of Continua) FD program.

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## LIMIT-EQUILIBRIUM TECHNIQUES

Limit-equilibrium methods are widely applied in slope-stability analysis, primarily because of their simplicity and ease of use. Of these methods, the method of slices used with a circular failure surface is often employed since circles are convenient to analyse and often approximate the observed failure surface. Unfortunately, there are many instances when the slip-circle method cannot be applied, since any *a priori* choice of the shape of the slip surface cannot be generally valid. This is especially so for non-homogeneous profiles in which a slip surface is expected to form along a line of minimum resistance such as bedding, anisotropy, etc. Observations of actual failure surfaces in the field indicate that slip surfaces can exhibit a variety of shapes, often with sharp transitions at the border between different materials. Thus, the case of a circular slip surface can be relegated to a special case of non-circular failure analysis in which the latter is used as the analytical technique.

The conventional limit-equilibrium analysis for a slope FOS involves two steps:

- (1) development of a rule of correspondence between a potential slip surface ( $X_i$ ) and the FOS; this functional relation is referred to as the safety function, where  $F_f$  is the functional value such that

$$FOS = F_f \{ (X_i) \}; \quad [1]$$

- (2) searching for the minimum FOS over all feasible slip surfaces; the surface ( $X'_i$ ), which represents this minimum value, is the critical slip surface such that

$$F_{\min} = \min F_f \{ (X_i) \} = F_f \{ (X'_i) \}, \quad [2]$$

where  $F_{\min}$  represents the minimum FOS described over a geometrical surface, with  $X_i$  describing the *i*th set of coordinates.

Considerable attention has already been given to the first aspect; the methods of Bishop<sup>1</sup>, Janbu<sup>2</sup>, Spencer<sup>3</sup>, and Morgenstern and Price<sup>4</sup> are well known. The second aspect, the location of critical slip surfaces, has been analysed by techniques based on variation calculus (Baker and Garber<sup>5</sup>, and Revilla and Castillo<sup>6</sup>). This technique is not particularly rigorous (Castillo and Luceño<sup>7</sup>), and numerical methods based on numerical programming techniques predominate today. Typical applications are described by Celestino and Duncan<sup>8</sup>, Nquyen<sup>9</sup>, Arai and Tagyo<sup>10</sup>, and Baker<sup>11</sup>.

Methods for the analysis of slope stability must be valid for any admissible slip surface and, in addition, rigorous in terms of whether or not all equations of equilibrium are satisfied for each slice of the slope, specifically the interslice resultant or shear force. For the location of any critical slip surface, the first requirement is that the rule of correspondence between a potential slip surface and the FOS method is, in fact, rigorous. The general Spencer method fulfils this criterion.

As noted previously, to locate the critical slip surface, minimization methods based on variational techniques can be used, except in cases where the slope profile consists of multiple layers; in that case, the derivatives of various quantities at layer boundaries are not well defined and the solution becomes complex.

The technique of numerical analysis, specifically that of the dynamic-programming method, can be used in conjunction with the Spencer method of slices. Such a procedure has been developed by Baker<sup>11</sup> and will be used here to illustrate the methodology of dynamic programming as applied to limit-equilibrium analysis and, later, to FE/FD stress analysis.

## THE SPENCER METHOD OF SLICES

Spencer<sup>12</sup> originally presented the slice-based technique for the rigorous analysis of circular slip surfaces, and later extended it to cover non-circular slip surfaces. Baker<sup>11</sup> generalized the method by including external loads in the analysis. An abridged derivation is presented here (after Baker), upon which the subsequent application of the minimization routine is based. Readers are referred to Spencer<sup>3,12</sup> for the complete analysis.

Consider a general slip surface as shown in Figure 1, divided into a number of discrete slices. Figure 2 shows the forces acting on any such slice. Without any assumptions regarding the interslice forces,  $Q_i$ , the problem is indeterminate. Spencer assumes that

- (1) all the interslice forces are parallel and act at an unknown angle  $\theta$
- (2) all the moments  $M_i^R$  and  $M_i^W$  are zero for slices  $i = 1$  to  $n$ ; thus, the forces  $N_i$  and  $T_i$  are assumed to act at the centre of the slice base and  $W_i$  through the centre of the slice.

In any limit-equilibrium method, the limiting condition specifying the relationship between the shear and the normal forces on the slice base is required. For the Spencer analysis, as with many other methods, the linear Mohr-Coulomb failure criterion is used, together with an idealized elasto-plastic stress-strain relationship, as shown

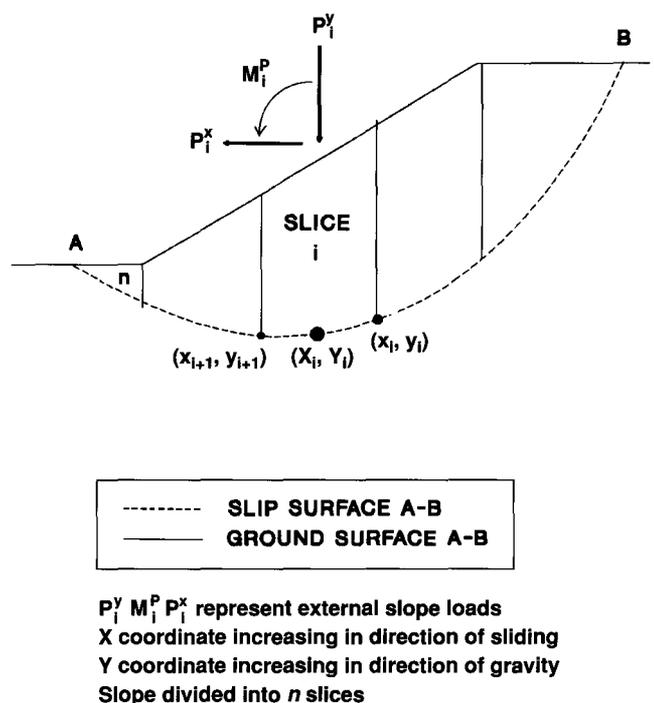
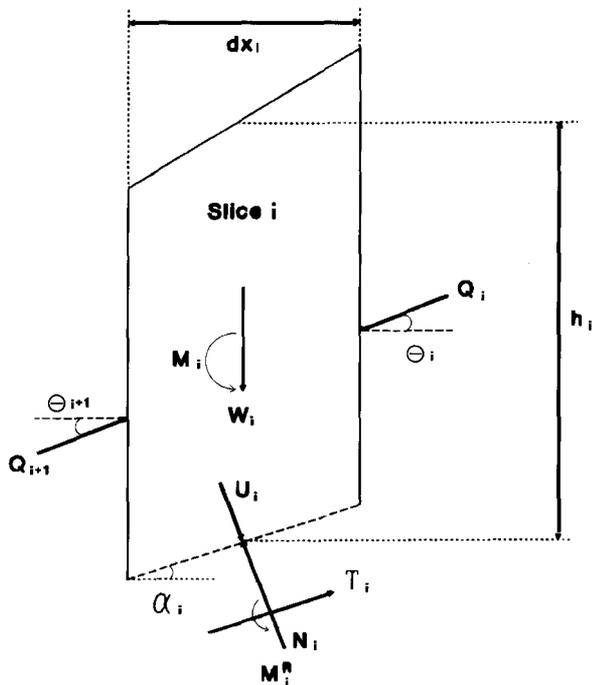


Figure 1—General slope geometry for Spencer-based slice analysis



For the *i*th slice,  
**W** = weight of slice  
**M** = moment of **W**  
**Q** = interslice force  
**θ** = angle of interslice force  
**M<sub>i</sub><sup>R</sup>** = moment of **N** and **T**

Figure 2—Definition of forces acting on a single slice

in Figure 3. For the Spencer analysis, in particular, this relationship is given by

$$\tau_i = C_i + (N_i - U_i) \tan \phi_i, \quad [3]$$

where

- $\tau_i$  = shear strength at base of slice
- $C_i$  = effective cohesive force
- $N_i$  = normal force on base of slice
- $U_i$  = pore-water force
- $\phi_i$  = effective angle of internal friction.

For a particular slip surface, the FOS must be equal to the FOS for overall moment equilibrium ( $F_m$ ) and the FOS for overall force equilibrium ( $F_f$ ). The FOS is thus obtained as a solution of the following equations:

$$FOS = F_f = F_{ff}(F_f, \theta) \quad [4]$$

$$FOS = F_m = F_{fm}(F_m, \theta), \quad [5]$$

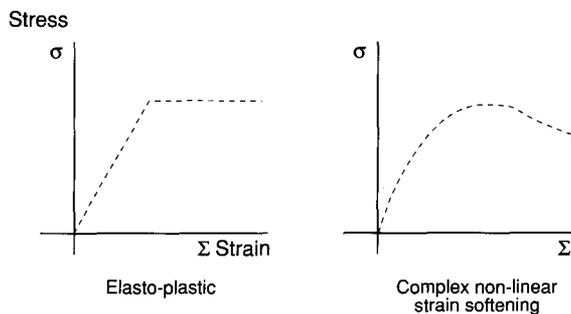
where  $F_{ff}$  and  $F_{fm}$  are both functionals of the slip surface as well as  $\theta$ .

Resolving horizontally and vertically, re-arranging, and substituting for  $\tau_i$  give

$$F_f = \frac{\sum_{i=1}^n (E_i / D_i)}{\sum_{i=1}^n (S_i / D_i)} \quad [6]$$

$$F_m = \frac{\sum_{i=1}^n (E_i / D_i) R_i}{\sum_{i=1}^n \left[ (S_i / D_i) R_i + (P_i^x h_i + M_i^p + M_i^w) \right]}, \quad [7]$$

### Constitutive relationships



### Yield relationships

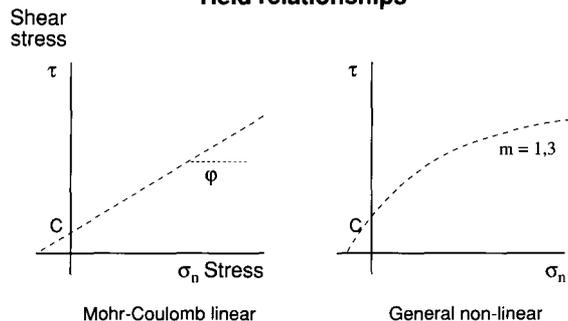


Figure 3—Typical linear and non-linear constitutive and yield relationships

where

$$E_i = k_i \{ C_i + \tan \theta_i [(P_i^y + W_i) \cos a_i - P_i^x \sin a_i - U_i] \}$$

$$S_i = (P_i^y + W_i) \sin a_i + P_i^x \cos a_i$$

$$D_i = \cos(a_i - \theta) \left[ 1 + \frac{k_i \tan \phi_i \tan(a_i - \theta)}{\tau} \right]$$

$$R_i = (Y_i - Y_c) \cos \theta - (X_i - X_c) \sin \theta$$

( $X_c, Y_c$ ) = the arbitrary coordinate point about which moments are taken

$\tau = F_f$  or  $F_m$  when  $D_i$  appears in equations [6] and [7] respectively.

The solution to these equations is reached iteratively. This then establishes the rule of correspondence between the slip surface and the FOS. To locate the position and shape of the surface that renders the lowest FOS for the slope, a dynamic programming technique is applied.

### DYNAMIC PROGRAMMING

Dynamic programming (DP) is a technique often used in the solving of multistage optimization problems. It is covered extensively by Dreyfus<sup>13</sup>, who illustrates the well-known application of the method to the optimization of transportation networks. For a thorough introduction to the theory and technique of DP, readers are referred to Larson and Casti<sup>14</sup>.

Baker<sup>11</sup> pioneered the application of DP to slope-stability problems. In that work, he developed a numerical algorithm for locating the position of the critical slip surface, using Spencer's modified technique of slices. His technique is

briefly described here as a precursor to an illustrative application of the method.

When the DP method is applied to a typical problem geometry, a number of stages ( $n + 1$ ) should be set up in the domain of the analysis, as shown in Figure 4. At any one of these stages, the position of the slip surface may be located at one of a finite number of states ( $j$ ) for each stage. If each stage is ascribed the notion of time, then, as time progresses, a trajectory is described through the stage-state domain. A value can be ascribed to each stage-state combination ( $DG_i$ ), known as the return function, based on the FOS described in equation [7]. As the trajectory develops, an optimal value function is defined ( $H_i [j]$ ), which describes the minimum value of the return function (i.e. the FOS) from the initial stage to the current location. This is described mathematically by the optimality principle of Bellman, such that

$$H_{i+1} [j] = \min_{j=1 \text{ to } T_i} [H_i(j) + DG_i(j, k)] \quad \begin{matrix} j=1 \text{ to } n \\ k=1 \text{ to } T_{i+1} \end{matrix} \quad [8]$$

where  $T_i$  represents the finite number of states at stage  $i$ .

Boundary conditions are applied so that the initial- and final-stage states, albeit unknown, are located at specific stages ( $i = 1$  and  $i = n + 1$  respectively) and the return function  $DG_i$  is then defined based on the FOS equation [6] or [7].

For theoretical reasons it is impossible to directly minimize equation [6] or [7], and an auxiliary function  $G$  is

used from which the return function  $DG_i$  is derived:

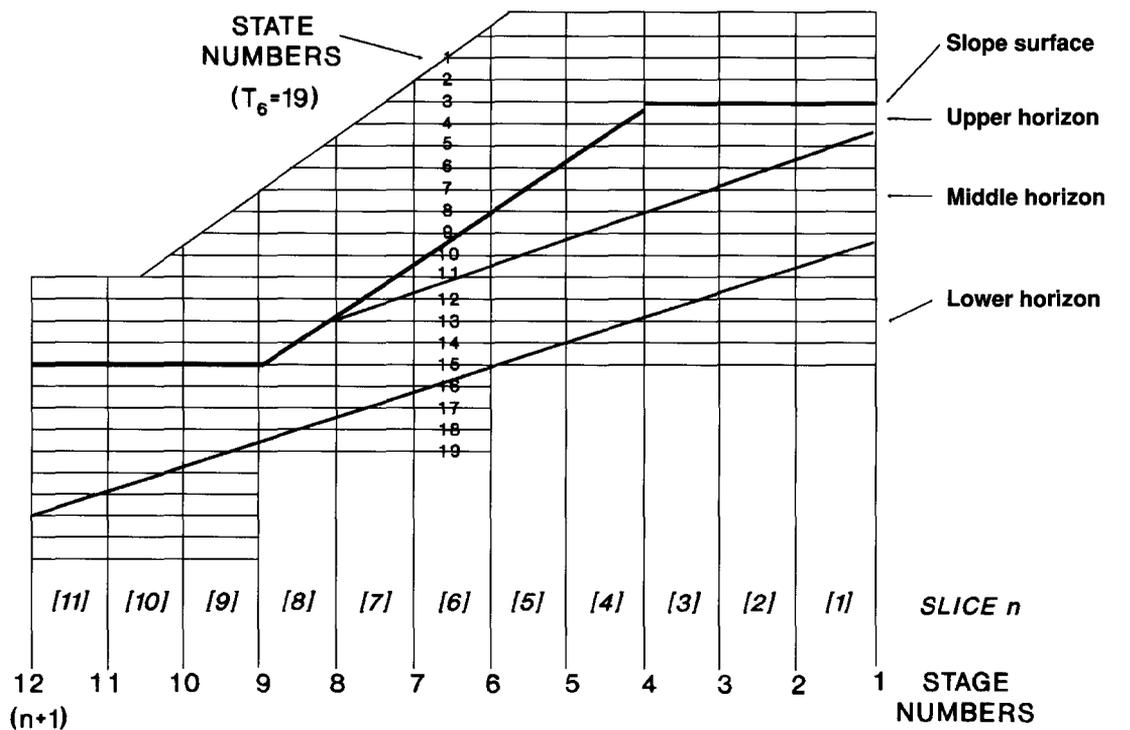
$$DG_i = R_i \frac{E_i - F_m S_i}{D_i} - F_m (P_i^x h_i + M_i^P + M_i^W) \quad [9]$$

The solution to the equation is found iteratively<sup>11</sup>. When equation [8] is applied to each state of each stage, the value of the minimum FOS is found at  $H_{n+1}(1)$ , and the shape of the critical slip surface is found from the recorded values of  $j$  corresponding to the minimum values of the brace [ ] at each stage state.

### TYPICAL SLOPE ANALYSIS

The hypothetical slope used for this particular analysis consists of three layers of material. For the purposes of this slope, pore pressures arising from the level of the phreatic surface are ignored (i.e. drained conditions). The layer parameters given in Table I form the basis of the Mohr-Coulomb linear failure criterion, equation [3].

From an analysis of the data given in Table I, it is reasonable to assume that any critical slip surface will form within only the upper and middle layers since the strength of the lower layer is such as to prohibit deep-seated failure. By the use of equations [6] and [7] modified to cater for the different unit weights of soil in each layer, the slope is assessed to find the location of the critical slip surface. The DP approach yields the absolute minimum value for the overall FOS and disregards any local minima that may



The search region outside the slope is used to determine the slip surface slope entrance and exit points.

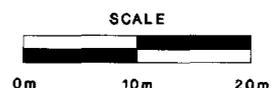


Figure 4—Slope discretization scheme in the DP technique (after Baker<sup>11</sup>)

**Table I**  
Soil strength parameters for the example problem

Horizon	Cohesion, <i>C</i> kPa	Friction angle, $\phi$ degree	Unit weight kN/m
Upper	30	15	19
Middle	10	7	19
Lower	100	30	19

exist. To this end, the problem was analysed layer-wise to illustrate the location of local (layer-restricted) minima.

The results of the analyses are shown in Figure 5 for critical slip surface locations and corresponding FOS in the upper and middle layers. A Bishop circular-failure analysis was also carried out within each layer, by a grid search routine to locate the slip circle associated with the lowest FOS. As can be seen from the results, the DP method gives a lower FOS for failure in each layer than the Bishop method, and the form of the critical slip surface is more appropriate to layered materials. Field tests and observations<sup>15,16</sup> confirm that failure surfaces exhibit sharp breaks between the borders of different materials, and are coincident with these borders for some portion of their length. The DP approach coupled with the Spencer modified-slice method would thus appear to provide reasonable solutions to the location of critical slip surfaces in slope-stability problems.

One of the drawbacks associated with this type of analysis and the limit-equilibrium approach in general is the inability of the method to easily cater for non-linear constitutive or failure relationships. The real behaviour of many materials is well modelled by a linear-failure method over a small range of normal stresses acting on the failure surface. However, the actual stresses along a failure surface are usually low<sup>17</sup>, and the failure envelope over the applicable stress region can exhibit considerable curvature, which will result in an over-estimation of the slope FOS. This is typically illustrated by Hoek<sup>18</sup>, who adopted an

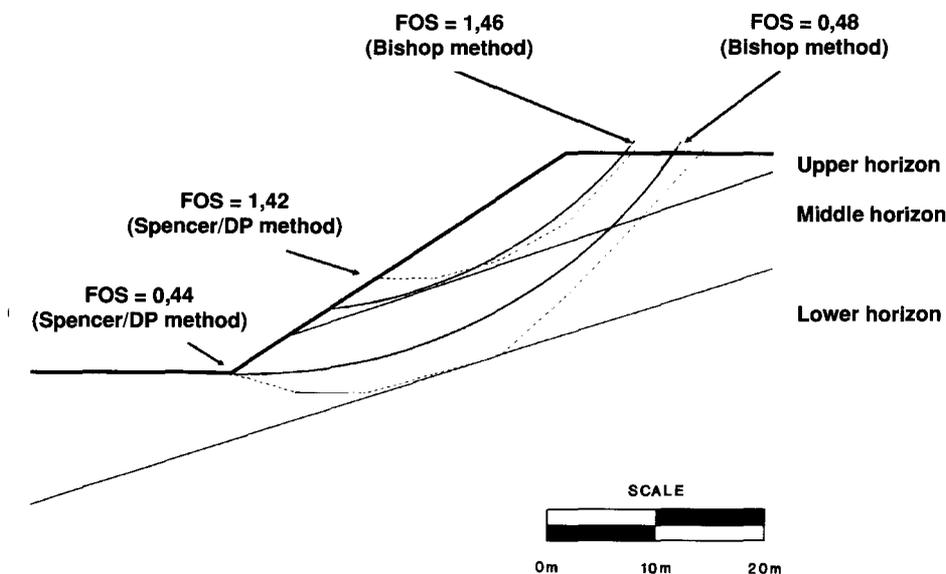
empirical non-linear failure criterion to more closely model the response of a failed waste dump, but applied the technique only to the Janbu method of slices in which the slip surface is assumed *a priori* to be circular.

The limitation imposed by the relationships between a material's constitutive and failure criteria can be overcome if the requirements of a limit-equilibrium analysis are combined with FE/FD techniques. With the latter techniques, a more representative multi-layer model can be analysed in conjunction with more complex constitutive and failure criteria relationships than allowed by the simplistic elastic-plastic Mohr-Coulomb limit-equilibrium approach. Such non-linear yield models are described by Hoek<sup>18</sup>, Zhang and Chen<sup>19</sup>, and Charles and Soares<sup>20</sup>. Although FE and FD methods can predict the overall stress-deformation behaviour of a slope more accurately, the absence of an overall FOS renders the method unsuitable for a classical stability analysis unless a method of locating the critical slip surface and associated minimum FOS is used.

### FINITE-DIFFERENCE SOLUTIONS

The FD program FLAC, together with a DP method similar to that described previously, is used here to illustrate the technique in which conventional FD stress-analysis data are used to generate an FOS and its associated critical slip surface.

The FLAC program, developed by ITASCA<sup>21</sup>, is an explicit finite-difference code simulating rock or soil structures that undergo plastic flow when their yield limit is reached. Materials are represented by two-dimensional grid elements which, in response to applied forces or restraints, follow a linear or non-linear stress-strain law. If the stresses are high enough to cause the material to yield and flow, the grid elements deform and move with the material represented by the grid. The Lagrangian calculation scheme used is well suited to the modelling of large distortions. In addition, the time-stepping approach to the solution of the equations of motions at each element node allows the



**Figure 5—Location of critical slip surfaces by the Spencer and DP techniques**

user to examine the development of yield (or material collapse) as it develops, instead of visualizing only the end state (equilibrium). For more details of the FD scheme and typical applications, readers are referred to ITASCA<sup>21</sup> and Cundall<sup>22</sup>.

The FD grid of elements representing the slope analysed in the previous section is shown in Figure 6, together with the corresponding three layers. The FLAC data sets used in the analyses are presented in the Addendum. This slope geometry is used here to illustrate the application of DP in conjunction with FD stresses to resolve the problem of minimum FOS and slip-surface location.

### DETERMINATION OF CRITICAL SLIP SURFACE FROM FLAC STRESS FIELD

As with the limit-equilibrium approach, it is necessary to develop a rule of correspondence between a potential slip surface  $AB$  of length  $l$  and the FOS. Following Yamagami and Ueta<sup>23</sup>, for a slip surface  $AB$  this can be defined as the ratio of available shear strength  $\tau$  to mobilized shear stress  $S$ , such that

$$FOS = \frac{\int_A^B \tau \, dl}{\int_A^B S \, dl} \quad [10]$$

When equation [10] is re-written in discretized linear form and the auxiliary function  $G$  is introduced to give the minimization function  $G_m$ ,

$$G_m = \min G = \sum_{i=1}^n (\tau_i - S_i FOS), \quad [11]$$

where  $\tau_i$  and  $S_i$  are taken to represent shear strength and mobilized shear force respectively. It then follows that the return function  $GD_i$  is given by

$$GD_i = (\tau_i - S_i FOS) \quad [12]$$

To evaluate the value of the return function between two stages  $i$  and  $i+1$ , the concept of either averaged or constant element stresses can be used. In the FLAC FD code, each element or quadrilateral is divided into four triangular elements, and the stresses are averaged over these four elements. For illustrative purposes and, since the model elements and thus the stress component changes between

are small, the simpler method of constant elemental stresses is used here.

From the resolution of the stress components  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  acting on an oblique plane (i.e. a line connecting any two stages  $i$  and  $i+1$ ) inclined at an angle  $\theta$  to the  $x$ -axis, the following expressions result:

$$\sigma = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - \tau_{xy} \sin\theta \cos\theta \quad [13]$$

$$\tau = \tau_{xy} (\sin^2\theta - \cos^2\theta) - (\sigma_y - \sigma_x) \sin\theta \cos\theta \quad [14]$$

Thus, the return function given by equation [12] can be evaluated once the elemental stress components as defined by equations [13] and [14] have been determined from the FLAC analysis. The summation is carried out over all the elements traversed by the trajectory. The available shear strength can be evaluated from the linear Mohr-Coulomb failure criterion:

$$\tau_i = \Sigma (C + \sigma \tan\phi) l, \quad [15]$$

or from any desired non-linear failure criterion. Such a criterion is given by the general relationship of Zhang and Chen<sup>19</sup>:

$$\tau_i = \Sigma C l \left[ 1 + \frac{\sigma}{\sigma_t} \right]^{1/m}, \quad [16]$$

where  $\sigma$  = normal stress acting on a failure surface

$\sigma_t$  = absolute value of tensile stress when  $\tau$  is zero

$m$  = non-linear coefficient.

When  $m = 1$ , equation [16] reduces to the Mohr-Coulomb linear criterion.

The mobilized shear stress can be evaluated from

$$S_i = \Sigma \tau l.$$

Boundary conditions are applied as described previously, and an appropriate number of states and stages is created in the domain of the problem to correspond to those elemental co-ordinates forming the stage and states required (Figure 4). The solution technique is identical to that described previously.

Since the value of the FOS in equation [12] is unknown, it is necessary to iterate to the correct minimum slip-surface location. This is achieved through summing the values for shear strength and mobilized shear force over the current

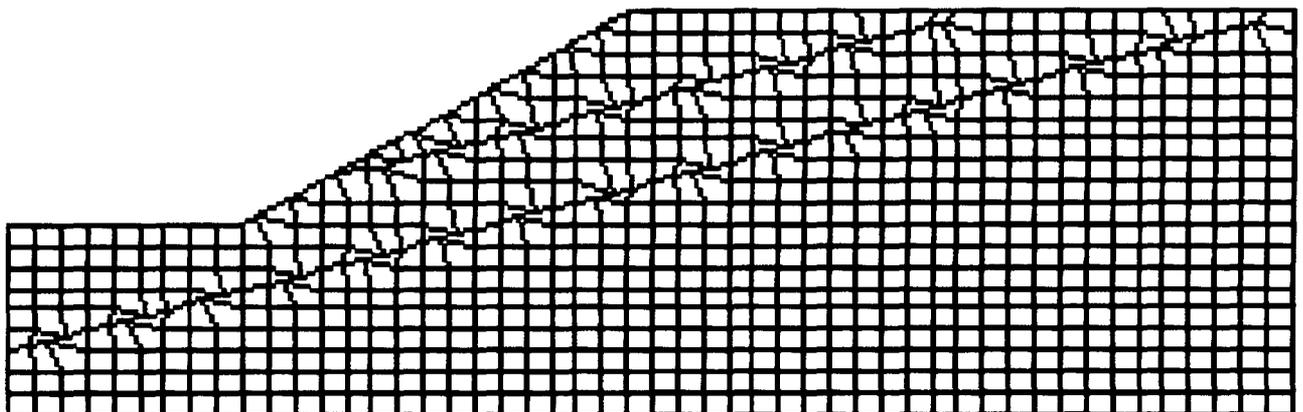


Figure 6—Original FD grid used in the modelling of slope and critical slip surface location

slip surface as found, and dividing to give a new estimate of FOS. If convergence is achieved within the specified tolerance, the critical slip surface is located; otherwise, a new value of FOS is calculated, and the minimization process is repeated.

### TYPICAL SLOPE ANALYSIS USING FD METHOD

The slope analysed previously is again used to illustrate the application of the FD technique coupled with the DP search routine to locate the critical slip surface. Two constitutive relationships are assessed in conjunction with two failure criteria:

- (i) a Mohr–Coulomb failure criterion with an elastic, perfectly plastic soil constitutive relationship (in plane strain with non-associated flow rule)
- (ii) a general non-linear failure criterion where cohesion and friction values reduce after the onset of plastic yield (a constitutive relationship modelling strain softening of the soil).

Criterion (i) is similar to that used for the limit-equilibrium analysis, whilst criterion (ii) models the non-linear effects of both the constitutive and the yield criteria. The choice of suitable non-linear criteria for modelling the real behaviour of a particular rock or soil type or, in this example, the choice of the coefficient  $m$ , is a subject in its own right, and readers are referred to Charles<sup>17</sup> and Charles and Soares<sup>20</sup> for a detailed discussion. Expressions describing a non-linear yield criterion are generally derived iteratively from the results of laboratory triaxial tests, and the transformation of these data into a non-linear yield model is well illustrated by Vermeer and De Borst<sup>24</sup>. The non-linear constitutive criterion is based on a hyperbolic

soil model in which non-linear elastic moduli are calculated as functions of confining pressure and mobilized strength. Readers are referred to Duncan and Chang<sup>25</sup> for details of the model, and to ITASCA<sup>21</sup> for the associated coding.

The FD technique is used to calculate the stress components generated in the mesh according to the *constitutive* relations specified in (i) and (ii). These data are then coupled with a DP search routine based on the *yield* relationships expressed in (i) and (ii), i.e. equations [15] and [16], to locate the critical slip surface and associated minimum FOS. These results are illustrated in Figure 7, which shows that these results are similar to those obtained by the Spencer method.

The ability of the FD technique to provide the engineer with more information concerning the behaviour of slopes during destabilization is illustrated in Figure 8. By use of the facility to update the mesh shape as failure proceeds, the resulting deformation of the elements and slope profile is seen.

Additionally, Figure 9 relates the horizontal deformation contours associated with failure along the interface between the middle and lower layers, with the maximum horizontal displacement occurring in the vicinity of the toe of the slope. The zone of displacement is seen to approximate that derived from the DP-based search routine for the critical slip surface. The location of the critical slip surface coincides with the region exhibiting the highest displacement gradient, as would be expected for a failed slope.

However, where the FOS of the slope is greater than unity, this effect is no longer so apparent, as witnessed by the absence of a second (shallow) failure region. In this case, recourse should be made to the velocity vectors representing the displacement. These are shown in Figure 10, confirming

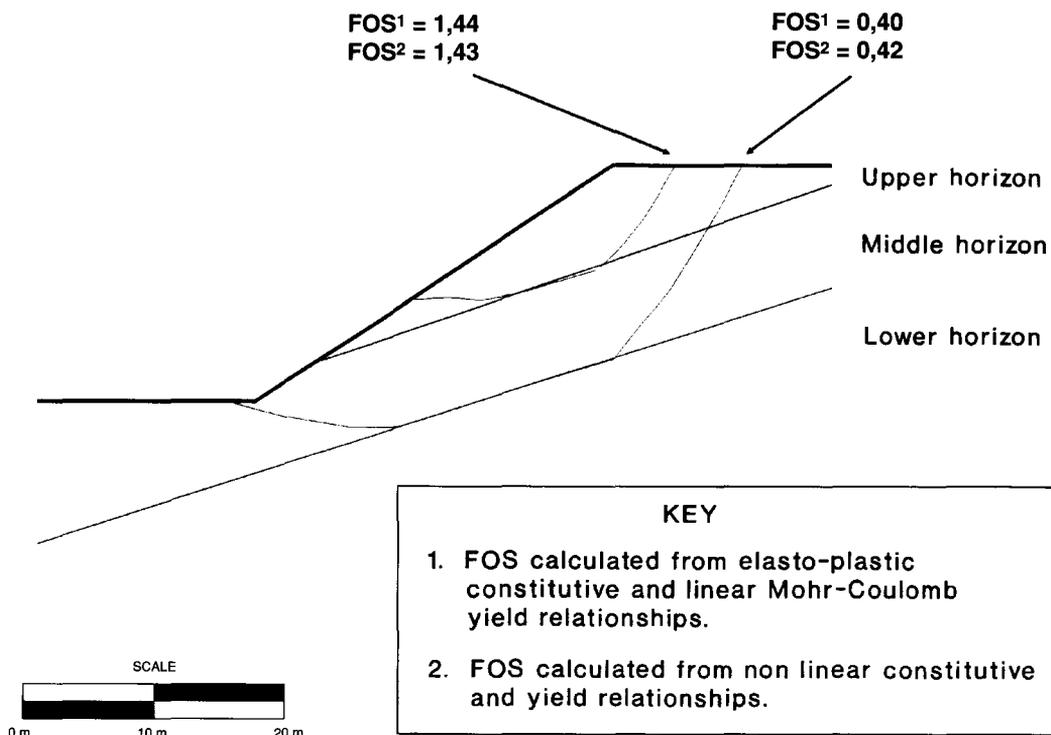


Figure 7—Location of the critical slip surface using FD and DP methods

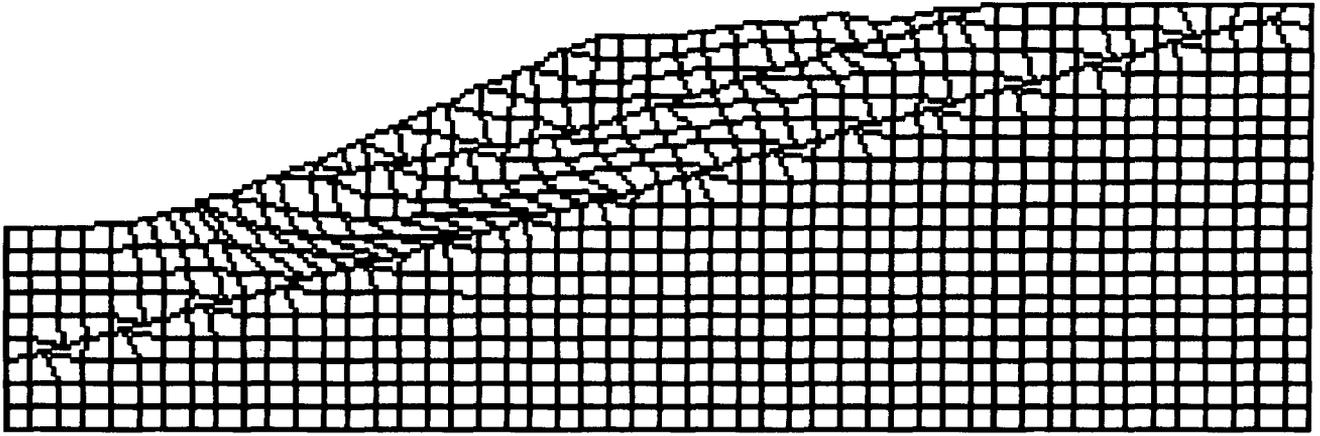


Figure 8—FD grid showing slope deformation as failure proceeds

the field observations of failure or movement of slopes coincident with layer boundaries.

Additionally, both deep and shallow failure surfaces are indicated. Figure 11 illustrates the contours of maximum shear-strain increment derived from slope displacements. This illustrates the shear-band formation associated with slope failure as would be seen in the field. Although the development and location of the band are sensitive to the model geometry, they nevertheless correlate well with the location of critical slip surfaces found by DP methods.

An indication of the current yield state of the materials comprising the slope is given in Figure 12. Since FLAC adopts a time-stepping approach to the solution of the slope model, the development of yield and the yield history of materials comprising the slope can be visualized. The variation of yield states within the slope and the critical areas (in terms of stability) are clearly seen. The facility to examine the yield history is an important consideration; areas of previously failed material, currently elastic, may exhibit significantly lower (residual) strength characteristics.

The advantage of this analytical technique over that of the conventional limit-equilibrium technique is realized through the extra data available to the engineer in terms of the progression of destabilization, and the attendant deformation and displacement. These results could be analysed and correlated with existing slope-monitoring data to predict and model the actual destabilization process more reliably, and thus predict the slope FOS more reliably.

It should be noted that the analytical technique adopted involves consolidation of the material prior to excavation under the action of gravitational forces. It is similarly feasible to model initial stresses in any material developed independently or as a result of sequential construction (e.g. embankments). Thus, some variation in the critical FOS is to be expected between the two models, although the correctness of any one particular technique (in terms of FOS and slip-surface location) cannot easily be assessed at this stage unless observed slope instabilities are modelled. One such exercise was undertaken by Talesnick and Baker<sup>26</sup>, who reported that a close approximation to the location of actual slip surfaces was achieved when the Spencer limit-equilibrium method was used in conjunction with the technique described for the location of critical slip surfaces.

## DISCUSSION AND CONCLUSIONS

The use of FD techniques within the framework of the limit-equilibrium approach has been illustrated, and the advantages of such a system have been expounded. However, the natural materials comprising a slope are inherently complex, and their constitutive and yield relations are generally simplified by the introduction of a certain degree of idealization. The simpler the relationships, the easier the analysis but the greater the deviation, in terms of behaviour, between the idealized and the real material. Thus, to strive for representative

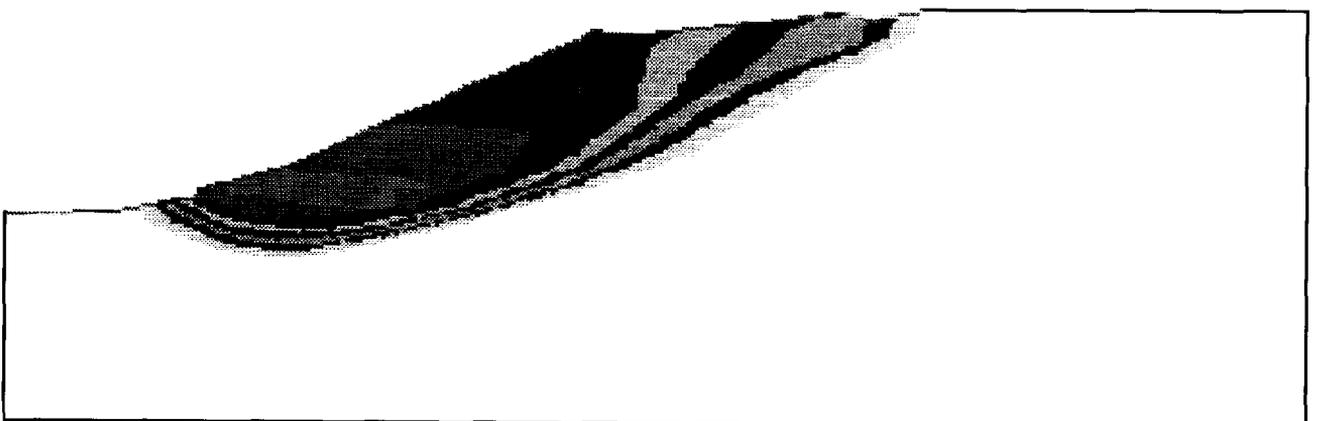


Figure 9—Contours of horizontal displacement during slope failure

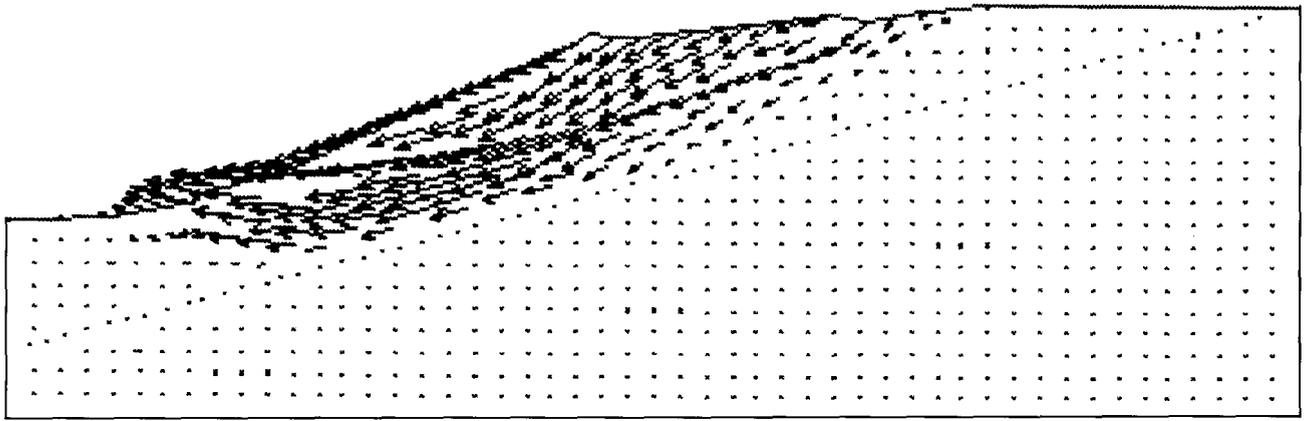


Figure 10—Velocity vectors representing the overall slope displacement during failure

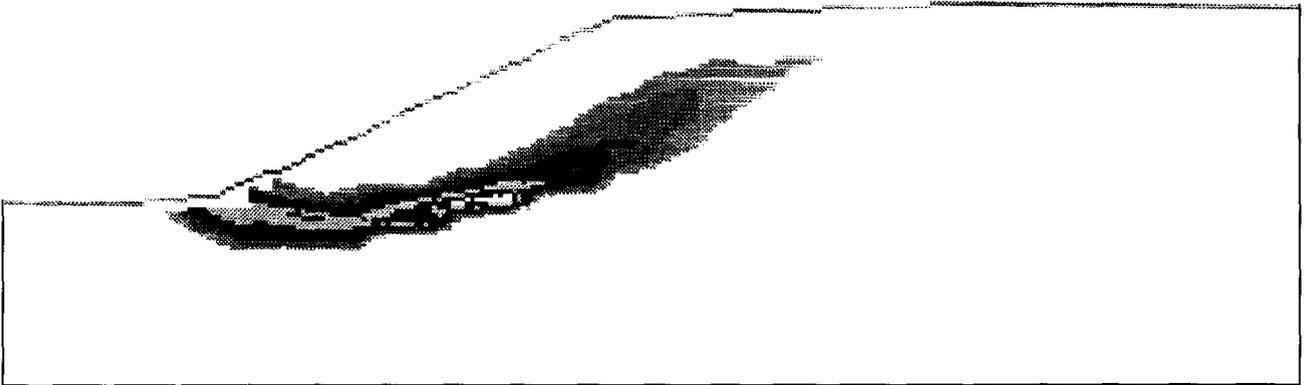
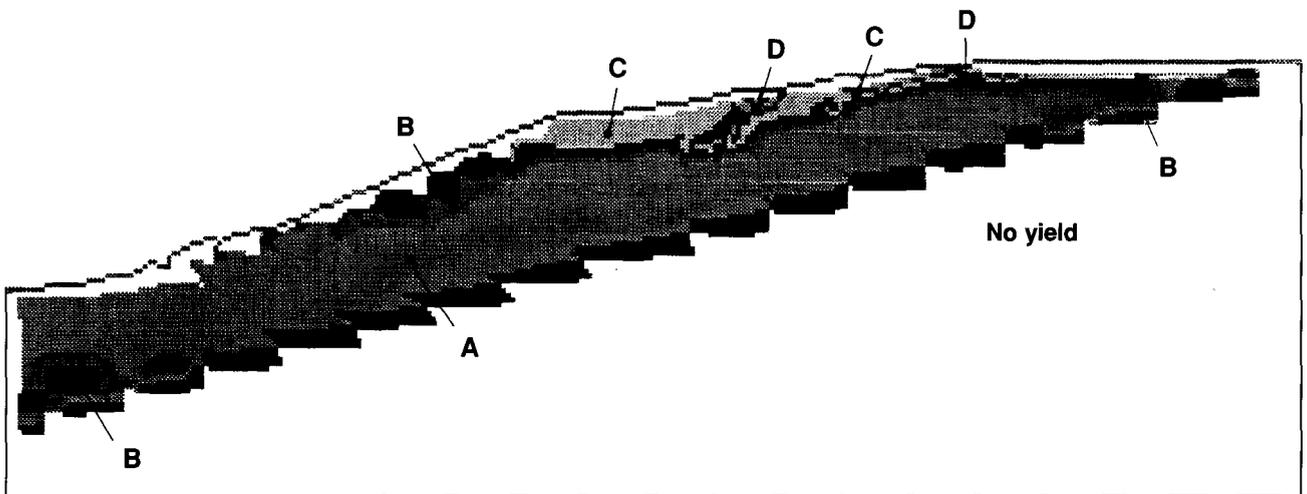


Figure 11—Formation of localized shear band (derived from shear-strain increments) associated with slope failure



- |                            |                             |
|----------------------------|-----------------------------|
| A At yield                 | B Elastic, at yield in past |
| C Uniaxial tensile failure | D Yield and tensile failure |

Figure 12—Current yield state and yield history of a slope as failure proceeds

relationships for the material in question would seem, given the widespread availability of computers, to be worth while. However, with the increasing complexity of the idealized material, the testing and analysis of all the required parameters may become onerous and expensive.

This then leads to the concept of balance of design, where the quality of the input data is matched with the degree of

analytical sophistication required to adequately predict the performance of a slope. From the foregoing, it is clear that the Spencer limit-equilibrium approach provides a good balance between the basic data requirements and the sophistication of the result. Through the use of an FE/FD method, more complex relationships can be modelled, but care must be taken not to assume that the corollary of an

increased level of sophistication in the analysis leads to an increase in the quality of the result, especially where the quality of the data has not improved.

Notwithstanding the above limitations, the FE/FD approach, coupled with a DP routine to locate the critical slip surface and associated FOS, would appear to give reasonable results. Additionally, it provides the engineer with a more complete picture of the stress conditions within a slope, accommodating either initial stresses within the slope or stresses developed as a result of construction or excavation. The option to fully specify a material's constitutive and yield relationships within the solution technique provides the engineer with an opportunity to evaluate the impact of such data on the overall slope stability, and to model displacement or inclinometer monitoring results. This enables the correlation between field observations and theoretically derived displacements to be established, and confirms the admissibility of the assumed failure geometry and model. Further work is nevertheless required to establish the benefit, in terms of the accuracy of the FOS and slip-surface location, of using the more refined approach offered by the FD technique. This could be realized through the back-analysis of a number of well-documented failures.

The technique has been shown to provide a detailed insight into the way that a slope will deform and fail, and therefore, coupled with the method illustrated for the location of critical slip surfaces, provides the engineer with a valuable additional method of analysing slope behaviour.

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## ADDENDUM

Coding for the various FLAC finite-difference slope models discussed is given below. For full details of the coding, data-file manipulation, and data-presentation techniques, readers are referred to ITASCA<sup>1</sup>.

- \* Initial grid definition, no slope excavated. Grid must be
- \* reasonably fine to allow good definition of shear bands

GRID 50,20

- \* Define model and parameters, initially elastic before slope
- \* excavated

MODEL ELASTIC

PROP S=30e6 B=100e6 D=1900

- \* Assign model boundaries and gravity

SET GRAV=9.81

FIX X i=1

FIX X i=51

FIX XY j=1

- \* Verification of model

HIS NSTEP=20

HIS UNBAL

HIS YDIS i=51 j=26

- \* Solve for initial stresses prior to excavation

SET FORCE=100

SOLVE

- \* Excavate slope profile

MARK i=1,10 j=10

MARK i=25,51 j=20

GEN LINE 18,18 48,38

MODEL NULL REG i=1 j=16

MARK i=1 j=4

MARK i=51 j=20

GEN LINE 0,6 100,38  
MARK i=13 j=12  
MARK i=39 j=20  
GEN LINE 24,22 76,38

- \* Typical slope analysis model 1 (El-Pl and M-C), strength parameters assigned to layers. Apply DP technique to model prior to assigning FLAC failure criteria and re-initialise displacement and velocity data

MODEL M REG i=26, j=18  
PROP s=30e6 b=100e6 c=30e3 f=15 d=1900  
reg i=26 j=18  
MODEL M reg i=26 j=14  
PROP s=30e6 b=100e6 c=10e3 f=7 d=1900  
reg i=26 j=14  
MODEL M REG i=26 j=6  
PROP s=30e6 b=100e6 c=100e3 f=30 d=1900  
reg i=26 j=6  
SET LARGE  
SOLVE

- \* Typical slope analysis model 2. Restart from excavated slope and apply DP technique to model prior to assigning FLAC yield criteria. Bulk and shear moduli are recalculated, based on Duncan and Cheng's model, for the complete slope (see ITASCA<sup>1</sup>).

MODEL ELASTIC  
CALL DUNCAN.FIS \*refer ITASCA<sup>1</sup> for full FISH routine  
SET PAT=1.054E6 K=450 ND=0.26 RF=0.8 KUR=1800

KB=280  
SET MD=0.19 COH=30E3 FRI=15 NSL=20 NSU=1  
NSUPS=100  
SUPSOLVE \*refer ITASCA<sup>1</sup> for full FISH routine

- \* Yield criteria are applied according to the ss model.
- \* Restart from excavated slope and reinitialise displacement and velocity data to visualise failure of slope

MODEL SS REG i=26, j=18  
PROP s=30e6 b=100e6 c=30e3 f=15 d=1900 ftab=1  
ctab=2 reg i=26 j=18  
MODEL SS REG i=26 j=14  
PROP s=30e6 b=100e6 c=10e3 f=7 d=1900 ftab=3 ctab=4  
reg i=26 j=14  
MODEL SS REG i=26 j=6  
PROP s=30e6 b=100e6 c=100e3 f=30 d=1900 ftab=5  
ctab=6 reg i=26 j=6  
TABLE 1 0,15 0.01,13.3 0.02,11.9 0.03,10.8  
TABLE 2 0,30e3 0.01,27.8e3 0.02,25.9e3 0.03,24.2e3  
TABLE 3 0,7 0.01,6.3 0.02,5.8 0.03,5.4  
TABLE 4 0,10e3 0.01,9e3 0.02,8.2e3 0.03,7.5e3  
TABLE 5 0,30 0.01,27.8 0.02,25.9 0.03,24.3  
TABLE 6 0,100e3 0.01,89e3 0.02,79e3 0.03,71e3  
SET LARGE  
SOLVE

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1. ITASCA. *FLAC v. 3.0-Fast Lagrangian analysis of continua*. Owner's handbook, ITASCA Consulting Group Inc., 1991.

## BOOK Review

### Rockbursts and seismicity in mines

Reviewer: C.M. Pretorius\*

*Rockbursts and seismicity in mines, Proceedings of the 2nd International Symposium on Rockbursts and Seismicity in Mines, Minneapolis, 8-10 June 1988. Edited by Charles Fairhurst, University of Minnesota, Minneapolis, Minn., USA. A.A. Balkema Rotterdam/Brookfield 1990. P O Box 1675, 3000 BR Rotterdam, Netherlands; Old Post Road, Brookfield, VT 05036, USA. (Price unknown).*

This book contains the proceedings of papers presented at the Second International Symposium on Rockbursts and Seismicity in Mines, held in Minneapolis in June 1988.

As rockbursts are considered to be one of the most serious problems and least understood phenomena associated with mining in highly stressed rock, several papers illustrate the different conditions under which fault slip event and crush-type rockbursts occur. Although the emphasis is mainly on hard rock mining, papers on coal mining bumping and seismicity related to gas fields are also published.

Different aspects related to rockbursts and seismicity in mines are addressed. In the first part of the proceedings different authors refer to the mechanics of seismic events and rockbursts, induced by, and related to mining

operations. It deals with different studies world-wide on the variety of conditions under which seismic activity have been observed and recorded. Modelling of rockburst and the nature of empirical engineering that are intended to reduce the incidence of damaging events are also shown by several authors.

Several papers describe technology involved in full seismic wave-form data acquisition, the advantages of such new digital systems, and the provision of data upon which some of the more fundamental issues in the field could be studied.

The last part of the book includes papers on rockburst control and techniques that could possibly be employed as to reduce the rockburst hazard and damage caused by it.

In general, one could say that there is a general trend away from seismic prediction techniques toward understanding the mechanics of preventive mining methods to minimize the rockburst hazard.

This book is recommended to seismologists and rock engineering practitioners dealing with seismicity. It should however not be seen as 'state-of-the-art' technology, as rapid advancement is being made in this field. It does however give information that is very useful in applied seismic engineering.

\* Regional Rock Engineer, Gengold.

## Belt-conveyor program\*

The Belt-conveyor program of Creative Engineering Co., now includes additional features, and is used worldwide by engineers who demand cost-efficient and trouble-free performance of any CEMA or ISO-5048 belt-conveyor configuration for the handling of bulk solids. It is easy to use, accepting both metric and imperial units, and is employed in feasibility studies, marketing, procurement, engineering, construction, and mine engineering; it is also used in the upgrading of existing conveyors, and in the designing of new systems. The details of the program are as follows:

- **HAULPLAN**, from an input of 49 variables, compares the haulage cost of quarry trucks versus belt conveyors plus in-pit crushing. The output includes internal rate-of-return, net-present-value, and profitability index, as well as cost per tonne. Mine engineers can quickly model different haulage plans to determine which is best. Using HAULPLAN, one company saved 15 million US dollars by avoiding an unwise investment. (Details are available on requests.)
- **DESIGN** features the simultaneous display of both cost and engineering data, providing at once all the information necessary for the user to achieve both proper design and low cost. Nine separate price multipliers ensure that the price structure displayed is correct for both currency and discount. The weakness and hidden costs of manual design or other software are avoided. The net payoff is many times the price of the program.
- **REVERSE ENGINEERING** is used by mining and bulk-material-handling engineers to identify defects in order to make corrections beforehand and so avoid costly downtime. Fatigue failure is a culprit that defies detection before disaster strikes. This program exclusively integrates the tension profile and component design with flashing caution warnings that cannot be ignored. This feature alone is saving one mining company more than US\$250 000 per year by eliminating lost production due to pulley breakage.
- **FAILURE** to meet breakaway requirements on starting can lead to electrical failure and 'abort start' disasters. This program resolves complex equations and warns with a simple caution.
- **SPILLAGE** cleanup is one of the most costly factors of maintenance. It often results from insufficient tension, cal-

culated to meet only drive needs, and fails to consider belt sag everywhere along the belt line during running, stopped, deceleration, and acceleration conditions. This program makes an automatic calculation of any extra tension needed, which is also vital to achieve the cost savings of a graduated idler space without incurring even greater spillage. Taking advantage of this analysis, one company salvaged excess idlers from a single conveyor for use elsewhere and saved US\$175 000.

- **VENDORS** of belt-conveyor services and components increase their market share by providing their customers with the results of this program as an added benefit. Timely service and instant price tabulation on the comprehensive printout helps develop new business in servicing, systems, belts, and parts. HAULPLAN identifies prospects having a genuine economic need, and then helps close sales with persuasive facts. The use of this program has become the prime marketing tool of more than one industrial supply firm. One such store in Montana (USA) sold US\$500 000 worth of belt, bearings, idlers, etc., for one job to a walk-in customer.
- **MANY OPTIONS** of data entry provide flexibility to accommodate any format in which those data are received. For example, rectangular coordinates are entered when survey coordinates are available to avoid common accumulative error. The writing of a conveyor file requires but a few minutes. The program accepts, for the international user, non-CEMA components. Optional continuum data entry accommodates multiple booster drives, horizontal curves, vertical curves, and two-way simultaneous conveying. Brakes, regenerative declines, and dual drives are readily calculated. The least-cost drive arrangement or idler series is easily determined. (It is never the cheapest, but which is it? This program can tell.)
- **CAUTIONS** with colour-coding interactively guide the user towards proper design and component selections to avoid operating an maintenance problems. The output is printed in an engineering, procurement, or marketing format. Also included are separate programs for the design of conveyor trusses, belts, trajectories, and curves to complete the all purpose 'tool box' needed to design cost-efficient belt conveyors of every configuration.
- **FREE DEMO** is available and can be used for the design of any conveyor up to 100 ft or 30 m in length, and for CEMA and other benchmark examples up to 40 000 ft or 12,192 m in length. Quick and easy demo instructions plus a disk-based 160-page instruction manual are included.

\* Issued by Creative Engineering Co., 3513 Century Drive, Bakersfield, CA 93306 1238, USA..



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