Cokriging and its effect on the estimation precision

by E. Yalçin*

Synopsis

The data set often contains one or more secondary variables. These secondary variables are usually spatially cross correlated with the primary variable. The addition of the cross-correlated information reduces the variance of the estimation error by using the cokriging method.

The grade estimation results of ordinary kriging and cokriging methods are evaluated and it was found that the estimation variance of cokriging method is slightly lower than ordinary kriging. The effect of variation of block dimension on the estimation results is also investigated and it was seen that the cokriging estimation variance values also decrease as the block dimension increases.

Introduction

Nowadays geostatistics cannot be ignored when dealing with resources evaluation. Geostatistics is a branch of applied statistics. The geostatistical tool used to describe and quantify the spatial variability of numerical variables is known as a 'variogram'. An internally consistent mathematical model describing the variograms of individual variables, as well as the cross-variograms between variables can be termed a linear model of coregionalization (LMC).

Kriging is an interpolation process that minimizes the estimation variance defined from a prior model for a covariance. Kriging calculates weights that result in optimal and unbiased estimates. Within a probabilistic framework, kriging attempts to minimize the error variance and systematically set the mean of the prediction errors to zero. Besides, it is known that the kriged estimates will be smoothed and would require some post-processing before being used as recoverable resources or reserves.

In general, the estimates are derived using only the sample values of one variable. However, a data set will often contain not only the primary variable of interest, but also one or more secondary variables. These secondary variables where spatially cross correlated with the primary variable can contain useful information about the primary variable. This information can be included within the estimation process via cokriging. It seems reasonable that the addition of the cross-correlated information contained in the secondary variable should help to further reduce the variance of the estimation error[1]. In addition, the inclusion of correlated data can also ensure the 'coherence' of the estimates.

As an example, consider the estimation of the elevation of the roof and floor of a coal-seam. If the roof and floor elevations are independently estimated, there are no guarantees that these surfaces (as estimated) will not cross, thus yielding negative thickness estimates for the coal. There are two alternatives accessible to us: one is to estimate the elevation of one of the surfaces (so we have to make a choice as to which may be a better estimate), as well as the thickness of the coal-seam, assuming that these variables are independent. The elevation of the unestimated surface can be derived by adding or subtracting (as appropriate) the seam thickness from the estimated elevation. Alternatively, we can consider developing a cokriging estimate for the roof and floor elevations together. The co-estimation of these two surfaces ensures that negative seam widths will not be encountered, thereby ensuring the coherence of the estimates

Cokriging is a technique of estimation that ensures that the value of a variable estimated in a point in space, on the basis of the neighbouring values of one or several other variables, is the best possible according to following statistical criteria:

- the absence of bias between the estimated value and the true one; 'globally' for all estimates together and also within grade categories of the estimitates, i.e. 'conditionally'.
- the minimization of the variance of the estimations.

* *Department of Mining Engineering, Dokuz Eylul University, Bornova/İzmir, Turkey.*

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Cokriging offers additional advantages over ordinary kriging. It involves the use of a secondary variable (covariate) that is cross correlated with the primary or sample variable of interest. The secondary variable is usually sampled more frequently and regularly, thus allowing estimation of unknown points using both variables 'globally' for the mean of all estimates but also 'conditionally' for the estimates within individually specified grade categories. This can aid in minimizing the error variance of the estimation[2].

There are some cases where the cokriging offers little advantages over kriging or some other simpler techniques. These situations occur mainly when[3]:

- the cross-variogram between the variable to be estimated and another variable is null. The two variables are mutually independent and the ordinary cokriging of the variable to be estimated can be simplified by withdrawing this other variable from the linear model of coregionalization (LMC)
- the variogram of the variable to be estimated is a pure 'nugget effect'. There are no inter-dependences between values of the same variable. The ordinary cokriging of the variable then just computes the arithmetical mean of the values of this variable in the neighbourhood, since all values of this variable have an identical weight. The others variables have null weights and can be dropped
- the LMC contains one basic model only, and all values of the variables are known simultaneously at the coordinates of the points taken into account in the neighbourhood. In this case, the ordinary cokriging is equivalent to the simpler ordinary kriging of the variable to be estimated, the weights of the other variables being null.

In these different cases, the estimation of a variable can be speeded up significantly if the LMC is modified according to the above remarks.

To perform cokriging, it is necessary to model not only the variograms of the primary and secondary data, but also the cross-variogram between the primary and secondary data. In this study, two-dimensional kriging and cokriging cross-validation grade estimation results of the sample points are compared by using the data taken from Küre copper mine in Turkey. Also, the effect of the block dimension on the estimation variance of cokriging technique is determined.

Formulation of cokriging

The matrix formulation of cokriging has been described by Journel and Huijbregths[4] and Myers[5,6]. The cokriging estimate is a linear combination of both primary and secondary data values and is given by the equation[7]:

$$\hat{u}_o = \sum_{i=1}^{n} a_i . u_i + \sum_{j=1}^{m} b_j . v_j \qquad [1]$$

where $\hat{u}_O$ is the estimate of U at location 0; $u_1,...,u_n$ are the primary data at n nearby locations; $v_1,...,v_m$ are the secondary data at m nearby locations; $a_1,...,a_n$ and $b_1,...,b_m$ are the cokriging weights that must be determined.

The development of the cokriging system is identical to the development of the ordinary kriging system. The definition of estimation error is:

$$R = \hat{U}_o - U_o = \sum_i^n a_i U_i + \sum_j^m b_i V_j - U_o \qquad [2]$$

where $U_1,...,U_n$ are the random variables representing the U phenomenon at the n nearby locations where U has been sampled and $V_1,...,V_m$ are the random variables representing at V phenomenon at the m nearby locations where V has been sampled.

An expression for the variance of the estimation error in terms of the cokriging weights and the covariances between the random variables are:

$$\begin{aligned} Var\{R\} = & \sum_i^n \sum_j^n a_i a_j Cov\{U_i U_j\} + \\ & \sum_i^m \sum_j^m b_i b_j Cov\{V_i V_j\} + 2 \\ & \sum_i^n \sum_j^m a_i b_j Cov\{U_i V_j\} - 2 \\ & \sum_i^n a_i Cov\{U_i U_o\} - 2 \\ & \sum_j^m b_j Cov\{V_j U_o\} + Cov\{U_o U_o\} \end{aligned} \qquad [3]$$

where $Cov\{U_iU_j\}$is the auto covariance between U_i and U_j, $Cov\{V_iV_j\}$ is the auto covariance between V_i and V_j and $Cov\{U_iV_j\}$is the cross-covariance between U_i and Vj.

The set of cokriging weights must satisfy two conditions. First, the weights must be such that the estimate given in Equation [1] is unbiased. Second, the weights must be such that the error variances given in Equation [3] are the smallest possible. One way of guaranteeing unbiasedness is to ensure that the weights in the first term sum to 1 while those in the second sum to 0[8,9]:

$$\sum_{i=1}^{n} a_1 = 1 \ and \ \sum_{j=1}^{m} b_j = 0 \qquad [4]$$

The Lagrange multiplier method may be used to minimize error variance with two constraints. To implement the method we simply equate each nonbias condition to 0, multiply by a Lagrange multiplier and add the result to Equation [3]. This gives the following expression:

$$\begin{aligned} Var\{R\} = & w^t C_z w + 2\mu_1 \\ & \left(\sum_{i=1}^{n} a_i - 1\right) + 2\mu_2 \left(\sum_{j=1}^{m} b_j\right) \end{aligned} \qquad [5]$$

where μ_1 and μ_2 are the Lagrange multipliers. The minimized error variance can be calculated using Equation [3] or it can be simplified by making substitutions using the Lagrange multipliers. The simplified version is:

$$\begin{aligned} Var\{R\} = & Cov\{U_o U_o\} + \mu_1 - \\ & \sum_{i=1}^{n} a_i Cov\{U_i U_o\} - \sum_{j=1}^{m} b_j Cov\{V_j U_o\} \end{aligned} \qquad [6]$$

Case study

The data used in the comparison of ordinary kriging and cokriging estimation results were taken from Küre-Aşiköy copper deposit. Figure 1 shows the location of the drill holes. The length weighted mean copper and sulphur grade values of 97 drill holes were calculated and the univariate statistical analysis results and histograms of copper and sulphur grades are given in Figures 2 and 3, respectively. As shown in the figures, the distribution of copper grade is lognormal, where the sulphur grade is nearly distributed normally.

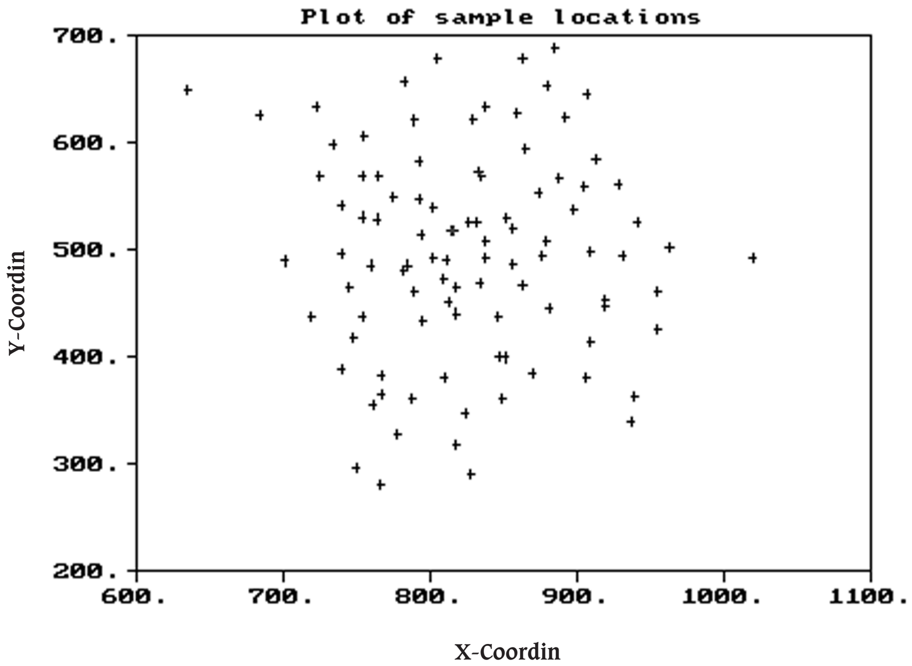

Figure 1—Location of the drill holes

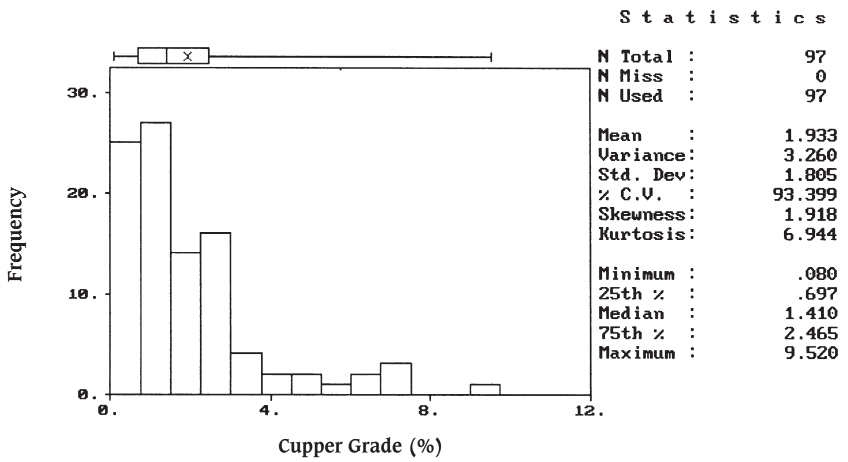

Figure 2—The histogram and statistical analysis result of copper grade

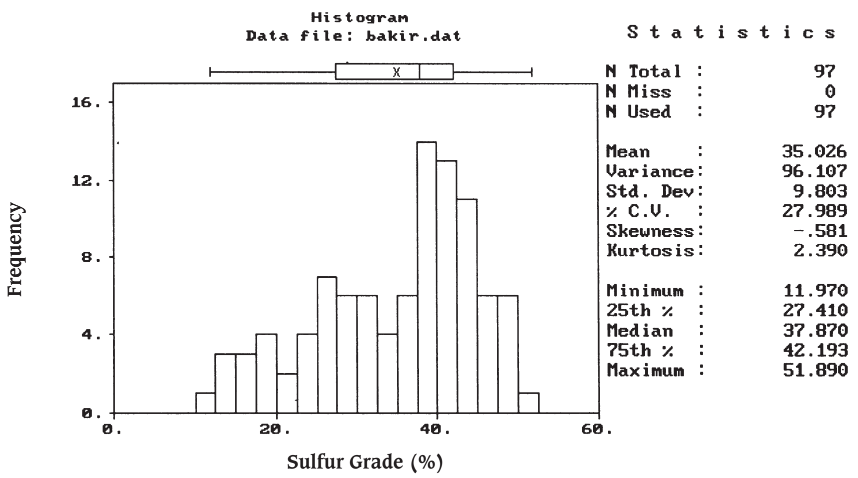

Figure 3—The histogram and statistical analysis result of sulphur grade

The correlation between copper and sulphur grades was determined with the linear correlation coefficient, which is one of the measures of correlation. It was found that the linear correlation coefficient is 0.454. It means that there is a positive correlation between copper and sulphur grades and the relationship is not too strong.

Variogram and cross-variogram analysis

In geostatistical studies, the fitting of the linear model of coregionalization (LMC) to direct and cross experimental semivariograms is usually performed with a weighted least-squares (WLS) procedure based on the number of pairs of observations at each lag[10]. Goulard and Voltz[11] describe an algorithm to fit an optimal linear model of coregionalization to a multivariate process.

The variogram analysis of copper and sulphur grades was carried out and mean variogram models whose angular tolerance is 90 degrees are given in Figure 4 and Figure 5. The mean variogram parameters found are given in Table I. The cross-variogram parameters between copper and sulphur pairs are also given in the same table. In Figure 6, the cross-variogram model is shown.

Ordinary kriging and cokriging estimation results

In order to compare the estimation results of ordinary and cokriging techniques, a cross validation technique that allows to compare estimated and true values using only information available in the sample data set was used. Cross validation results are most commonly used simply to compare the distributions of the estimation errors from different estimation procedures[7].

Table I

Variogram and cross-variogram parameters

Variable	Nugget , C_0	Sill, C	Range, a	Model type
Copper	1.6	2.0	90	Spherical
Sulphur	40	60	70	Spherical
Copper+Sulphur	5.0	2.9	70	Spherical

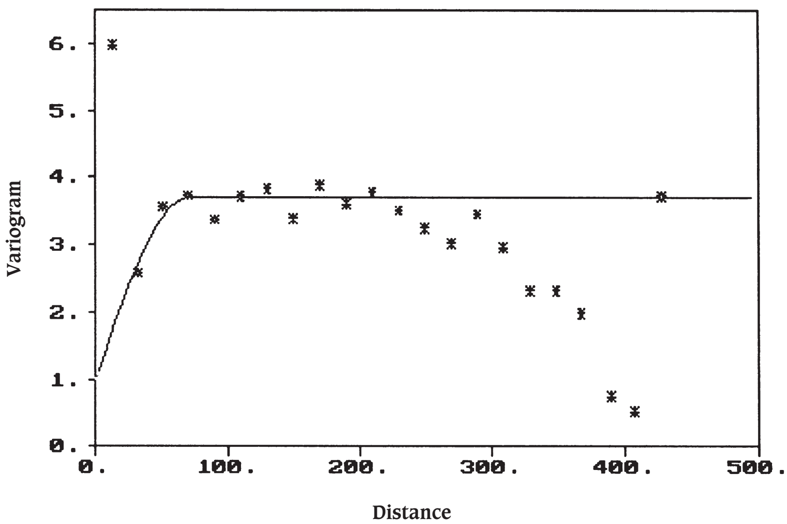

Figure 4—Copper grade variogram model

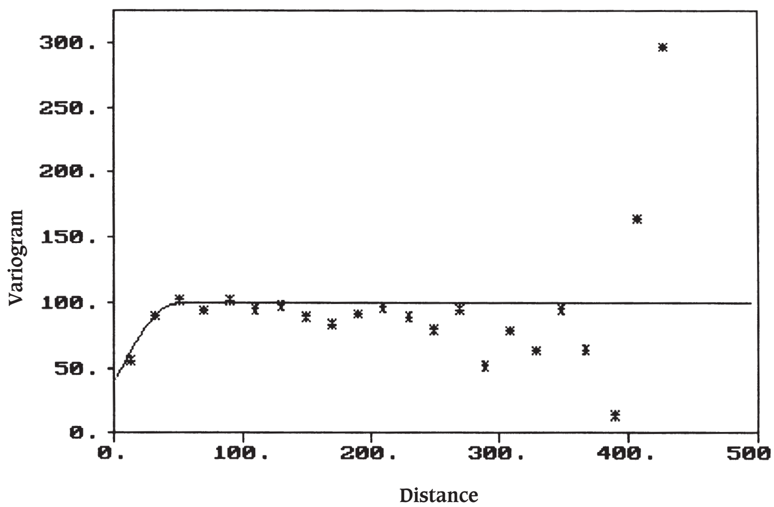

Figure 5—Sulphur grade variogram model

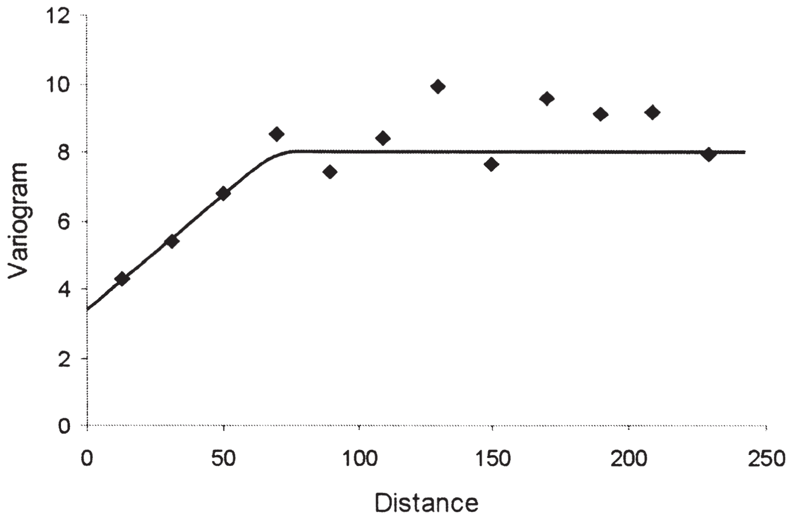

Figure 6—Cross-variogram model

Table II

Ordinary kriging and cokriging estimation results for copper

	N	Mean, %	Standard Deviation	Minimum	Maximum
True grade	97	1.93	1.84	0.08	9.52
Ordinary kriging technique					
Estimated grade	92	1.96	1.40	0.08	8.84
Kriging variance	92	1.94	1.01	0.03	6.46
Error	92	0.03	2.01	-6.85	5.98
Absolute error	92	1.38	1.46	0.02	6.85
Correlation of the estimates and true grades: Slope: 0.19 Intercept: 1.59 Corr. Coeff: 0.25					
Cokriging technique					
Estimated grade	92	1.94	1.41	0.08	8.84
Kriging variance	92	1.86	0.89	0.03	4.31
Error	92	0.02	2.02	-6.85	5.98
Absolute error	92	1.38	1.48	0.02	6.85

Correlation of the estimates and true grades: Slope: 0.25 Intercept: 1.35 Corr. Coeff: 0.35

Copper grade of sample points was estimated by using the ordinary kriging and cokriging techniques. The kriging search radius was taken as 60 metres that was found from cross-validation analysis and the variogram parameter values given in Table I were used in the estimations. The minimum number of samples used for estimation is taken as 2, excluding the value of the estimation location. This search routine is sparse and yields results which are conditionally biased (see slopes in Table III). However, the main objective of this study is to demonstrate the relative advantages of cokriging over ordinary kriging and this should not be affected significantly by the presence of such biases in both types of estimates. Detailed analyses of the effects of the search radius and routine on these biases, including the need when data is limited to search beyond the variogram range(s), as well as the need to eliminate these biases for purposes of detailed mine planning and selective mining, can be found in references 12 and 13. The univariate statistical analysis results of estimation results of both methods are given in Table II.

As shown in Table II, the mean estimated copper grade of the ordinary kriging technique is higher than the actual mean grade, whereas the cokriging estimate is close. In the case of the kriging variance, the estimate from the ordinary kriging technique is also slightly higher than that of the cokriging technique. The mean error of cokriging is lower than that of ordinary kriging. The estimation of copper grades by using the cokriging technique that gives better estimation results than the ordinary kriging technique.

Effects of block dimensions and cokriging radius on the cokriging estimation

The degree of accuracy of an estimate depends on several factors such as the size of the blocks being estimated and the spacing, configuration, and number of samples used for estimation. The effect of block dimension on the estimation quality of cokriging technique was investigated by increasing the block size. The same kriging plan described earlier was used as the search routine. For the same reasons as given

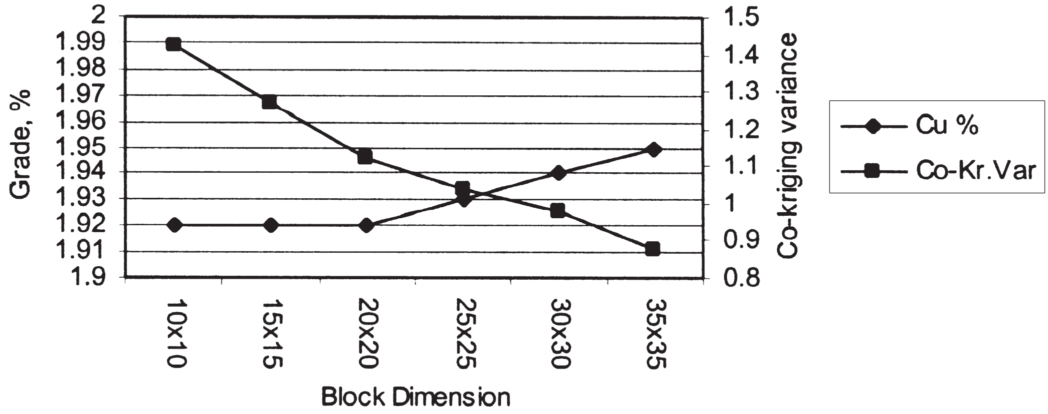

Figure 7—Variation of mean grade and estimation variance values with block dimensions

Table III

The copper grade estimation results at different block dimensions

Block dimension m x m	Ore block number	Mean	
		Cu %	Cokriging estimation variance
10 x 10	1121	1.92	1.42
15 x 15	504	1.92	1.27
20 x 20	281	1.92	1.12
25 x 25	184	1.93	1.04
30 x 30	129	1.94	0.98
35 x 35	96	1.95	0.88

earlier, these block estimates could also be conditionally biased, but this should also not affect the objective of showing the decrease in the error variances as the block size increases. In Table III, the estimated mean copper grade and cokriging estimation variance values are given for the block dimensions from 10 m × 10 m to 35 m × 35 m.

For the copper grade estimations given in Table III, the mean grades of block dimensions 10 m × 10 m, 15 m × 15 m and 20 m × 20 m are same. After the 25 m × 25 m block dimension, it starts to increase as the block dimension increases. In the case of the 35 m × 35 m block dimension, only 96 blocks are estimated; therefore, the grade estimation results are not reliable. The cokriging estimation variance values decrease as the block dimension increases as shown in Figure 7.

Conclusions

Because geostatistical estimation provides both an estimate and variance of the error associated with the estimate, a tendency exists to think that it gives the final word on evaluation of a deposit. The estimated grade values are used as input values in the feasibility study for mine planning, so the variance of error should be as low as possible to minimize the risk in the feasibility study.

In this article, the estimation results of ordinary kriging and cokriging techniques are compared and it was seen that the cokriging technique estimates the variable with lower estimation variance than the ordinary kriging. Because of the low correlation coeficient (0.454) between copper and sulphur grades, the improvement of the relative kriging standard deviation for cokriging is not high.

It was also seen that the cokriging estimation variance decreases as the block dimension increases.

As a conclusion, it can be said that, if the secondary variables are present or available, then the use of these secondary variables via the cokriging tecnique could be advantageous.

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