Variance propagation in toll smelting operations treating multiple concentrate stockpiles

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Synopsis

Metallurgical accounting for toll treatment smelters pose a number of challenges. Multiple feed stockpiles have to be accounted for as part of the monthly recovery estimation. Moreover, multiple metallurgical intermediates make up part of the in-process inventory, which also influence the monthly recovery calculation. Errors propagate from the measurements of volumes, assays, moisture fractions and bulk densities in the various material types through multiple steps up to the point where the final recovery is calculated. These errors contribute non-linearly to the variance in the final recovery estimate. This paper develops the mathematical formulation for variance propagation in toll smelting operations, including the effect of in-process inventory, assuming measurement biases have been eliminated beforehand. Operational data from a Southern African copper smelter is used for a case study.

The method of propagation of variance showed that uncertainties in stockpile assays were the main contributors to variance in the recovery estimate. Variance in the volume and bulk density uncertainties contributed a secondary, yet significant, proportion to the overall recovery variance. It was determined, for the given case study, that the recovery variance depended on the calculation method used and that variance propagation via the two-product formula was smaller than recovery variances calculated via the standard recovery formula. However, the probability that the two product formula will give inaccurate (versus imprecise) results is significantly more due to the practical difficulty of equiprobable sampling of tailings streams from smelter plants (which includes materials such as slags and flue dust). Recommendations are made on how to achieve a reduction in overall uncertainty for toll treatment smelters.

Keywords: metallurgical accounting, copper smelters, pyrometallurgy, variance propagation, errors, precision, sampling

Introduction

Improvements in metallurgical accounting systems are often motivated by the drive to improve corporate governance and transparency. Many non-ferrous smelter managers are confronted with widely varying recovery estimates, depending on the calculation method used and the range of possible bases from which recovery is estimated. This has led to recovery being a poorly defined, but highly significant number ('key process indicator') bandied about by metallurgists, managers and investors, all with a different perception as to what is really meant.

In this paper it is shown how variance propagates through plant material balances. As stockpiles of raw materials, products and inventory always fall within the battery limits of a smelter plant they make up part of the working capital, a figure directly reflected on the balance sheet of each smelter. The cumulative effect of uncertainty in the measurement of stockpile volumes, moisture levels and bulk densities propagates non-linearly through to the variance in average plant feed assays and estimated recoveries. There is little published information on what precision can be ascribed to the copper recovery from concentrate to anode or blister over a given metallurgical accounting period.

Where the mine and the smelter are an integrated unit, the overall recovery from mine to metal is of primary importance and changes can be made either at the mill or the smelter to maximize the overall recovery. Where the smelter recovery is extremely important is when concentrates are purchased or tolled because of the contract structures where the payable metal values mean that the smelter must achieve a certain minimum recovery in order to break even.

This paper is an attempt to quantify the variance in the recovery from first principles using plant data where the accounting principles are based on a 'check in-check out' system (Wills, 2005) which is often used for pay-metal accounting on a smelter. However, the 'check in-check out' system becomes impractical at smelters for nonpay elements and volatile elements such as sulphur (also elements such as arsenic, antimony, zinc)

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where bulk quantities are not transported via a checkpoint and continuous and variable losses occur via flue gases and fugitive emissions (such as from Pierce Smith converters, and molten matte transport in ladles).

Various calculation strategies are now considered for a range of typical metal accounting approaches. In the discussion below, bias (due to gross or systematic measurement error) has been ignored. In other words, all listed plant measurements are assumed to be unbiased. A number of methods exist for bias detection, such as the 'cusums method' (Bartlett, 2005) for a lumped, plant-wide test, or through statistical tests (François-Bongarçon, 2005) on individual sampling points using paired t-tests. Should biases be found to be present, they should be eliminated through proper calibration, maintenance, or repair (such as of leaks) as required. Sometimes total redesign of the sampling system is required according to the principles of representative sampling.

Recovery calculation from assays only: application of the two-product formula

If the weights of monthly smelter feed, blister copper (product) and discard slag (tails) are $\overline{F}$, P and $\overline{T}$ respectively and their corresponding assays $\bar{f}$, p, and $\bar{t}$ then :

$$\overline{F} = P + \overline{T} \quad [1]$$

i.e. : total material input = total material output, over the period of a month, and

$$\overline{F} \cdot \bar{f} = P \cdot p + \overline{T} \cdot \bar{t} \quad [2]$$

i.e. : the valuable metal, in this case copper, is balanced.

The bar above the feed and tailings variables indicate representative quantity over the monthly period, with inclusion of stockpiles for all feed or tailings material types.

It follows that:

$$\overline{F} \cdot \bar{f} = P \cdot p + \left(\overline{F} - P\right) \cdot \bar{t} \quad [3]$$

which gives the split fraction to the product stream as:

$$\frac{P}{\overline{F}} = \frac{\left(\bar{f} - \bar{t}\right)}{\left(p - \bar{t}\right)} \quad [4]$$

Subsequently, the metal recovery (Cu in this case) can be defined as :

$$R = \frac{p \cdot P}{\bar{f} \cdot \overline{F}} \cdot 100\% \text{ or :} \quad [5]$$

$$R = \frac{p \cdot \left(f - \bar{t}\right)}{\bar{f} \cdot \left(p - \bar{t}\right)} \cdot 100\% \quad [6]$$

The variance in any calculated amount (say recovery, or calculated stockpile mass, or average concentration) can be found from the variances in the measured quantities (such as assays, volumes, measured weights, etc.) through the summation of the squared partial derivatives of the calculated variable with respect to the independent measured quantities each multiplied with the respective variance in the measured quantity. The mathematical proof can be found in many a good text on mathematical statistics, such as Kreyszig (1970) and is based on the Taylor series expansion of the error of a multivariable function. This relationship is usually referred to as the 'propagation of variance' or the 'Law of propagation of error (LPE)'. This paper applies this principle to smelter plants treating multiple material types, where stockpiles and in-process inventories are present.

Xiao and Vien (2003) have noted that the addition of the second Taylor series term may be required in cases of large relative standard deviations of measurements. They state that the law of propagation of error (variance) is only an approximation, because second- and higher-order derivatives are neglected. For the non-linear variance propagation case, the precision of the approximation increases with decreasing relative variance, i.e. for small relative errors. Therefore, the truncation of the Taylor series approximation after the first term becomes less accurate when double digit relative standard deviations are measured on the plant, such as often found for slags and dusts (Eksteen *et al.*, 2004).

Additionally, it is assumed that the measurements are independent, which may not be the case in practice. The variance of a calculated variable (such as recovery) is a summation of the first derivative squared multiplied by the respective variances if the measurement variables are independent. For correlated measured variables, the covariances should be included in the error propagation estimate of recovery variance. Xiao and Vien (2003) did not consider the effect of stockpiles, but did show that Monte Carlo based methods estimated recovery variances to be of the same order but slightly larger than the variance propagation methods. Moreover, the Monte Carlo based methods allows one to see the effects of multiple normal and/or non-normal measurement distributions on the recovery distribution. As long as knowledge of the shape of the distribution is well defined, the Monte Carlo method is superior to variance-propagation. However, this knowledge is often not available, or can be obtained only at great cost, and plant metallurgists are often not inclined to perform Monte Carlo simulations on their desktop computer when a simpler formula-based estimation of variance is available.

The variance in plant recovery, R, based on the derived two- product formula (Equation [6]), can be found from its derivatives as follows :

$$V_R = \left(\frac{\partial R}{\partial \bar{f}}\right)^2_{p,\bar{t}} V_{\bar{f}} + \left(\frac{\partial R}{\partial p}\right)^2_{\bar{f},\bar{t}} V_p + \left(\frac{\partial R}{\partial \bar{t}}\right)^2_{p,\bar{f}} V_{\bar{t}} \quad [7]$$

where V_R, $V_{\bar{f}}$, $V_{\bar{t}}$ and V_p are the variances in R, $\bar{f}$, $\bar{t}$ and p respectively (for the month). For the two-product formula, the derived propagated variance in recovery in term of assays and assay variances only, is:

$$V_R = \left[\begin{array}{l} \dfrac{p^2 \cdot t^2}{\bar{f}^2} V_{\bar{f}} + \dfrac{\left(\bar{f} - \bar{t}\right)^2 \cdot \bar{t}^2}{\left(p - \bar{t}\right)^2} V_p + \\ \dfrac{\left(p - \bar{f}\right)^2 \cdot p^2}{\left(p - \bar{t}\right)^2} V_{\bar{t}} \end{array}\right] \cdot \frac{100^2}{\bar{f}^2 \left(p - \bar{t}\right)^2} \quad [8]$$

Recovery calculation from masses and assays: Traditional recovery estimate (excluding inventory)

As copper recovery is defined by Equation [5]: R $\frac{p \cdot P}{\bar{f} \cdot \overline{F}}$ = 100%

where P, p, $\overline{F}$, and $\bar{f}$ are defined as in Equations [1] and [2].

Then, as $R = f(F, P, f, p)$, the variance of R can be found from the partial derivatives with respect to each variable:

$$V_R = \left(\frac{\partial R}{\partial \bar{f}}\right)^2_{\bar{F},p,P} V_{\bar{f}} + \left(\frac{\partial R}{\partial \bar{F}}\right)^2_{\bar{f},p,P} V_{\bar{F}} + \left(\frac{\partial R}{\partial p}\right)^2_{\bar{f},\bar{F},p} V_p + \left(\frac{\partial R}{\partial P}\right)^2_{p,\bar{f},\bar{F}} V_P \quad [9]$$

where V_R, $V_{\bar{f}}$, $V_{\bar{F}}$, V_P and V_p are the variances of R, $\bar{F}$, $\bar{f}$, P and p respectively. If the derivatives are evaluated Equation [9] becomes:

$$V_R = \begin{bmatrix} \left(\frac{-p \cdot P}{\bar{F} \cdot \bar{f}^2}\right)^2 V_{\bar{f}} + \left(\frac{-p \cdot P}{\bar{F}^2 \cdot \bar{f}}\right)^2 \\ V_{\bar{F}} + \left(\frac{P}{\bar{F} \cdot \bar{f}}\right)^2 V_p + \left(\frac{P}{\bar{F} \cdot \bar{f}}\right)^2 V_P \end{bmatrix} \cdot 100^2 \quad [10]$$

Recovery calculation from masses, assays and inventory change

When inventory changes are to be incorporated, over and above new metal bearing material, the copper recovery is defined as follows :

$$R = \frac{p \cdot P}{\bar{f} + \bar{F} + \sum_i \Delta m_i a_i} \quad [11]$$

where m_i and a_i represent the change in copper of inventory item i based on opening (°) and closing (^{c}) masses and assays respectively, and the bar superscripts referring to actual monthly feed assay and net weight (all material types) fed to the smelter. Note that the there are different inventory material types to feed material types.

The net change in in-process inventory, per material type i, between month end and beginning is:

$$\Delta m_i a_i = (m_i a_i)^0 - (m_i a_i)^c \quad [12]$$

The associated variance in copper mass in inventory stockpiles over a month is:

$$V_{\Delta m_i a_i} = \left[\frac{\partial(\Delta m_i a_i)}{\partial a_i^0}\right]^2_{m_i^0, m_i^c, a_i^c} V_{a_i^0} + \left[\frac{\partial(\Delta m_i a_i)}{\partial a_i^c}\right]^2_{m_i^0, m_i^c, a_i^0} V_{a_i^c} + \left[\frac{\partial(\Delta m_i a_i)}{\partial m_i^0}\right]^2_{a_i^0, m_i^c, a_i^c} V_{m_i^0} + \left[\frac{\partial(\Delta m_i a_i)}{\partial m_i^c}\right]^2_{m_i^0, m_i^0, a_i^c} V_{m_i^c} \quad [13]$$

The sum of the variances for the total inventory change (all material types) is then the sum of the variances for the individual items (as they are statistically independent), i.e.:

$$V_{\Delta m_T} = \sum_i V_{\Delta m_i a_i} \quad [14]$$

Estimation of variances in feed masses and assays each per material type

The calculation for each material treated in the smelter is in two parts:

- calculation of opening and closing stock variance
- calculation of receipts variance.

The net change in mass of ore material/concentrate type k for a one-month period is:

$$\Delta F_k = M_{dk}^r + M_k^0 - M_k^c \quad [15]$$

The variance in feed mass is simply the sum of the opening and closing stockpile variances and the receipts variance, as the variances in opening, closing, and receipt masses are statistically independent (However, independence is questionable if there is a bias in stock measurement):

$$V_{\Delta Fk} = V_{M_{dk}^r} + V_{M_k^0} + V_{M_k^c} \quad [16]$$

where the superscripts o, c and r refer to the opening stocks, closing stocks and receipts respectively.

Opening and closing stock variance per material type

The mass of a given stockpile (opening or closing) is defined by the following equation :

$$M_k = Vol_k BD_k (1 - mf_k) \quad [17]$$

where M_k, Vol_k and BD_k are the dry weight, volume and bulk density stockpile of material type k, and mf_k is the moisture fraction of material k (% moisture divided by 100).

The variance of the dry weight of stockpile M_k is defined by the following equation :

$$V_{Mk} = \left(\frac{\partial M_k}{\partial Vol_k}\right)^2_{BD_k, mf_k} V_{Vol_k} + \left(\frac{\partial M_k}{\partial BD_k}\right)^2_{Vol_k, mf_k} V_{BD_k} + \left(\frac{\partial M_k}{\partial mf_k}\right)^2_{BD_k, Vol_k} V_{mf_i} \quad [18]$$

Then from the partial differentials:

$$V_{Mk} = \left[BD_k(1 - mf_k)\right]^2 V_{Vol_k} + \left[Vol_k(1 - mf_k)\right]^2 V_{BD_k} + \left[Vol_k BD_k\right]^2 V_{mf_k} \quad [19]$$

Receipts variance per material type

The dry weight of a shipment of material k is represented by the Equation [20].

$$M_{dk}^r = (1 - mf_k) \cdot M_{wk}^r \quad [20]$$

where M_{dk}^r is the dry weight of the shipment, M_{wk}^r is the as-received (wet weight) as measured on a weigh bridge and mf_k is the moisture fraction as previously defined. The corresponding variance of the dry weight of the receipt is defined by the Equation 21.

$$V_{M_{dk}^r} = \left(\frac{\partial M_{dk}^r}{\partial M_{wk}^r}\right)^2_{mf_k} V_{M_{wk}^r} + \left(\frac{\partial M_{dk}^r}{\partial mf_k}\right)^2_{M_{wk}^r} V_{mf_k} \quad [21]$$

Then from the partial differentials :

$$V_{M_{dk}^r} = (1 - mf_k)^2 \cdot V_{M_{wk}^r} + (M_{wk}^r)^2 \cdot V_{mf_i} \quad [22]$$

Feed assay variance in overall feed (all material types)

Average assay and associated variances for one material type over month duration

The check in-check out system of accounting defines the average feed assay f_k for one material type as follows :

$$f_k = \frac{a_k^r M_{dk}^r + a_k^o M_k^o - a_k^c M_k^c}{M_{dk}^r + M_k^o - M_k^c} = \frac{\text{nett mass of copper fed of material type } k \text{ during a month}}{\text{total mass of material type } k \text{ during a month}} \quad [23]$$

where M_{dk}^r, M_k^o and M_k^c are the dry tonnages for receipts, opening and closing stocks respectively and a_k^r, a_k^o and a_k^c are the receipt assays, opening assays and closing assays for copper respectively. This gives the average copper assay, per ore or concentrate type, over the duration of one month, taking into account receipt and the open and closing stockpiles.

The variance of the average assay is then:

$$V_{fk} = \left(\frac{\partial f_k}{\partial a_k^r}\right)^2 V_{a_k^r} + \left(\frac{\partial f_k}{\partial a_k^o}\right)^2 V_{a_k^o} + \left(\frac{\partial f_k}{\partial a_k^c}\right)^2 V_{a_k^c} + \left(\frac{\partial f_k}{\partial M_{dk}^r}\right)^2 V_{M_{dk}^r} + \left(\frac{\partial f_k}{\partial M_k^o}\right)^2 V_{M_k^o} + \left(\frac{\partial f_k}{\partial M_k^c}\right)^2 V_{M_k^c} \quad [24]$$

where V_{Mdk}^r, $V_{M_k^o}$ and $V_{M_k^c}$ are the variances of M_{dk}^r, M_k^o and M_k^c respectively and $V_{a_k^r}$, $V_{a_k^o}$, $Vakc$ are the variances of a_k^r, a_k^o and a_k^c respectively.

The partial differentials with respect to mass and assay are illustrated by the following :

$$\left(\frac{\partial f_k}{\partial a_k^r}\right) = \frac{M_{dk}^r}{M_{dk}^r + M_k^o - M_k^c} \quad [25]$$

and similarly through partial derivation with respect to M_k^o and M_k^c.

$$\left(\frac{\partial f_k}{\partial M_{dk}^r}\right) = \frac{a_k^r \cdot \left(M_k^o - M_k^c\right) - \left(a_k^o M_k^o - a_k^c M_k^c\right)}{\left(M_{dk}^r + M_k^o - M_k^c\right)} \quad [26]$$

and similarly through partial derivation with respect to a_k^o and a_k^c.

Average assay for a blend of multiple materials

A copper smelter usually treats more than one concentrate type and in some cases a variety of mattes and purchased scrap as well. The monthly average assay, as fed to the furnace (i.e. the equivalent of all stockpiles combined and fed via belt conveyor into the furnace over a one-month period) can be defined as follows :

$$\bar{f} = \frac{\sum_k f_k \Delta F_k}{\sum_k \Delta F_k} \quad [27]$$

The total mass fed from new metal-bearing material for the month (total mass excluding inventory change) is:

$$\bar{F} = \sum_k \Delta F_k \quad [28]$$

The variance of the average feed copper is defined as follows :

$$V_{\bar{f}} = \sum_k \left(\frac{\partial \bar{f}}{\partial f_k}\right)^2 V_{fk} + \sum_k \left(\frac{\partial \bar{f}}{\partial \Delta F_k}\right)^2 V_{\Delta F_k} \quad [29]$$

The partial differentials with respect to mass and assay are illustrated by the following :

$$\left(\frac{\partial \bar{f}}{\partial f_k}\right) = \frac{\Delta F_k}{\sum_k \Delta F_k} \quad [30]$$

$$\left(\frac{\partial \bar{f}}{\partial \Delta F_k}\right) = \frac{f_k \sum_{k=1}^n \Delta F_k - \sum_{k=1}^n f_k \Delta F_k}{\left(\sum_{k=1}^n \Delta F_k\right)^2} \quad [31]$$

for a total of k=1..n material types (concentrates fed to smelter).

So that the variance in the average feed assay, including the effect of stockpiles is given by :

$$V_{\bar{f}} = \sum_k \left(\frac{\Delta F_k}{\sum_k \Delta F_k}\right)^2 V_{fk} + \sum_k \left(\frac{f_k \sum_{k=1}^n \Delta F_k - \sum_{k=1}^n f_k \Delta F_k}{\left(\sum_{k=1}^n \Delta F_k\right)^2}\right)^2 V_{\Delta F_k} \quad [32]$$

Product weight variance

In the case considered here the product blister is weighed on an assized weighbridge with an accuracy of ±20 kg in 100 tons. Any weight measurement on this bridge has a relative standard deviation (RSD) of 0.01. Assuming a normal distribution of measurement uncertainties, the standard error can be said to be ±1.96xσ, within a 95% confidence interval (Kreyszig, 1970):

$$\text{where : } \sigma = \sqrt{\frac{\sum_{J=1}^{NS} (m = \bar{m})^2}{n-1}} \quad [33]$$

and *NS* is the number of weight determinations during calibration.

Product assay variance

The RSD was calculated by analysis of assay results from multiple spoon samples taken from the same ladle during blister copper pouring. The RSD was determined as 0.047.

Discard slag assay variance

Although the RSD for copper in discard slag was not directly measured, work done by Eksteen *et al.* (2004) as well as within the AMIRA P754 project suggests that the RSD is likely to be in the range 15–20% for metal/matte elements, where the element/species can be entrained (e.g. Cu in this case). The RSD is much smaller for dissolved metal losses than for entrained metal losses in slags. Due to the difficulty of achieving equiprobable sampling for melts, there is a significant risk of introducing bias due to the sampling process. This bias is also difficult to assess as finding the 'true' value is not economically viable. There is therefore a very good reason for using feed and product assays only, rather than including waste materials (such as slags and dusts). However, the inclusion of waste does aid in material balancing (using data reconciliation), but only if the composition can be determined without significant bias.

Worked example

The base data for the following cases is July 2005 actual data for a Southern African copper smelter A. For confidentiality reasons the sources of the materials treated are not given.

Recovery and variance calculation from assays using the two-product formula

Actual input data

$\bar{f}$: average copper assay in feed mix : 32.95%
p: copper assay in blister copper : 98.85%
t: copper assay in discard slag : 1.30%
$V_{\bar{f}}$: variance in average % copper in feed mix : 0.58
Vp: variance in % copper in blister copper : 0.0021
V_t: variance in % copper in discard slag : 0.0676

Then by substitution in Equation [6]:

$R = 97.32\%$

and using Equation [7]: $V_R = 147.18\ V_f + 1.65\ V_p + 43572.40\ V_t$

which on substitution with the values given above gives:

$V_R = 0.008 + 0.0000003 + 0.295 = 0.303$

and therefore the $\sigma_R = 0.55$, and the standard error with 95% confidence limits = ±1.1%

From this calculation the recovery at the 95% confidence level will be 97.3% ± 1.1%.

In this case the recovery variance is dominated by the variance in copper in the discard slag with a small contribution coming from the variance in the average copper in feed mix.

Recovery and variance calculation from masses and assays

Actual input data

$\bar{f}$: average copper assay in feed mix: 32.95%
p: copper assay in blister copper: 98.85%
t: copper assay in discard slag: 1.30%
$V_{\bar{f}}$: variance in average % copper in feed mix: 0.58
V_p: variance in % copper in blister copper: 0.0021
V_t: variance in % copper in discard slag: 0.0676
$\bar{F}$: total dry tons of feed: 8103 mt
P: blister copper produced: 2703 mt
$V_{\bar{F}}$: variance in the weight of feed: 11 771 mt²
V_P: variance in the weight of blister copper: 0.0729 mt²

Then by substitution :

$R = 100.1\%$

and $V_R = 0.00015\ V_F + 0.0014\ V_P + 92265.5\ V_f + 10251.7\ V_p$

$V_R = 1.80 + 0.0001 + 5.34 + 0.002 = 7.14$

and therefore $\sigma_R = 2.67$

From this calculation the recovery at 95% confidence limits is: $R = 100.1\% \pm 5.2\%$

In this case the variance in the recovery is significantly influenced by the feed assay variance and less so by the feed mass variance. The calculated recovery is not meaningful from a practical perspective, because it does not yet take inventory change into account.

Recovery calculation from masses, assays and inventory change

Input data

$\bar{f}$: average copper assay in feed mix: 32.95%
p: copper assay in blister copper: 98.85%
t: copper assay in discard slag: 1.30%
$V_{\bar{f}}$: variance in average % copper in feed mix: 0.58
V_p: variance in % copper in blister copper: 0.0021
V_t: variance in % copper in discard slag: 0.0676
$\bar{F}$: dry tons of feed: 8 103 mt
P: blister copper produced: 2 703 mt
$V_{\bar{F}}$: variance in the weight of feed: 11 771 mt²
V_P: variance in the weight of blister copper: 0.0729 mt²
$\sum_i \Delta m_i a_i$ = copper in inventory change: 80 mt

$V_{\sum_i \Delta m_i a_i}$: variance in copper in inventory change : 219 mt²

Then by substitution :

$R = 97.2\%$

and $V_R = 0.0116\ V_{\bar{F}} + 0.0013\ V_P + 286.3 + 9663.9\ V_p + 0.0353\ V_{\sum_i \Delta m_i a_i}$

$VR = 1.6 + 0.00009 + 4.72 + 0.002 + 0.27 = 6.59$

and $\sigma_R = 2.6$

For this case the inventory variance has some significance but the feed assay variance is still dominant.

Discussion of results

This work was undertaken to try and understand why the recovery numbers and unaccounted losses in smelter A varied by up to 5% in any month accounting period when the process streams and recycles were accounted for and the flow sheet was closed except for discard slag.

While the calculation of recovery variance from assays (two-product formula) gives good numerical results, it is certainly the least accurate because of the difficulties in practice of estimating the tailings assay variance, which in this case is discard slag. Moreover, due to the difficulty of implementing equiprobable sampling practice for melts, there is a significant probability of introducing bias through systematic error due to poor sampling. The calculation is relatively insensitive to feed assay variance mostly due to the totally overpowering effect of tailings variance.

It is preferable to go back to the calculations based on mass and assays including change in inventory, which are inherently more accurate, because reasonable estimates can be put around the basic parameters. The recovery calculation in this case does not rely on any variances associated with weights or assays of discard slag where the sampling technique is paramount and furnace conditions need to be stable at all times for the discard slag assay to mean anything.

In the case of the mass estimates, the variances in stock pile volumes, bulk densities and moisture levels can be established in the field. In the case of assays, the sampling and analytical errors are reasonably well known. The major contributor to the feed assay variance is the assay variability between lots and within lots, which comes back to the

performance of the concentrator at the mine. The initial assumption in this analysis was that the variance in copper assay between lots of nominally 400 tons would be representative of the variance within a lot. This was checked using 10 sequential lots of the individual concentrates and 10 wagons within a lot. The results are shown in Table I.

Table II is a summary of sensitivities against the base case as calculated using masses, assays and inventory change.

The five cases in Table II are as follows:

- The Base case is the actual case as observed for a given month for smelter A
- Case 1 assumes that the copper assay variance for concentrate A was applicable for all concentrates
- Case 2 assumes that all concentrates received during the month were treated during the month
- Case 3 is the 'best' case where there are no opening and closing stocks and the assay variance for concentrate A applies to all concentrates
- Case 4 assumes that there are opening and closing stocks for all concentrates treated during the month.

In all cases the average feed assay variance is the major contributor to the recovery variance although the mass variance becomes significant where there are opening and closing stocks for each concentrate type. As for all stockpiles, managing between empty piles is a practical way to control this problem. Piles which almost never empty are convenient —especially for hiding problems.

In all cases the contributions of variance of both the product weight (blister copper) and the product assay were insignificant, due to the high degree of precision.

Table I

Concentrate assays and measured: Standard deviations

Material type	Lot average Cu %	σ	Wagon average Cu %	σ
Concentrate A	23.70	0.92	23.97	1.48
Concentrate B	51.65	3.10		
Concentrate C	29.39	1.10		
Concentrate D	21.76	2.70	20.50	3.10
Concentrate E	30.84	2.70	31.62	3.00

Conclusions

The following conclusions could be drawn from the analyses presented above.

- A variance propagation model has been formulated for a copper toll treatment smelter. This variance propagation model took into account changes in stockpiles, as well as changes in inventories.
- It was determined that, despite the two-product formula being more precise (i.e. smaller variance in recovery estimate), it is inherently less accurate, due to challenges associated with the sampling of melts, dusts and gases in cases when these become the 'tailings', or waste of a smelter.
- The assay variance in feedstocks was always the largest contributor to variance in recovery, and was always greater than the contribution due to variances in net masses fed (which includes stockpile and inventory changes).
- The large observed variances in recoveries could be explained through the variance propagation model.
- The actual observed case (base case) showed that the standard deviation in the recovery estimate was 2.6%. The standard error in the monthly recovery estimate is therefore +/- 1.96 x 2.6% = ±5.1% with 95% confidence.
- There is therefore a clear business case to improve sampling practice and instrumentation on this specific toll treatment smelter. However, the most effective way to reduce the recovery variance is to reduce the assay variability between stockpiles. Subsequent to the reduction in assay variability, is the reduction in the variability in stockpile masses due to variability in volumes, bulk densities and moisture fractions. Good control over blending might go a long way to achieving this goal, i.e. variation between material type assays can be reduced through proper blending based on known assays of all material types being blended. However, doing this well usually requires a major capital investment.
- Bulk density variation can be reduced through the use of a narrow particle size distribution; volume variances can be reduced through improved surveying

Table II

Contributions to variance: comparison against base case

Comp.	Base case	% Cont.	Case 1	% Cont.	Case 2	% Cont.	Case 3	% Cont.	Case 4	%
$V_{\bar{F}}$	1.60	24	1.60	42	0.04	1	0.04	2	2.43	29
V_P	0.00009	0	0.00009	0	0.00009	0	0.00009	0	0.00009	0
$V_{\bar{f}}$	4.72	72	1.90	50	4.32	93	1.80	85	5.63	68
V_p	0.002	0	0.002	0	0.002	0	0.002	0	0.002	0
$V_{\sum_i \Delta m_i \beta_i}$	0.27	4	0.27	7	0.27	6	0.27	13	0.27	3
V_R	6.6		3.8		4.6		2.1		8.3	
σ_R	2.6		1.9		2.2		1.5		2.9	

techniques; and the variance in moisture fractions can be reduced by, for example, shielding the stockpiles from environmental changes in moisture (e.g. using a shed).

- As the variance in recovery is known, it serves as a baseline for developing improved material balance closure (data reconciliation) techniques, such as least squares (Narasimhan and Jordache, 2000; Romagnoli and Sánchez, 2000) or maximum likelihood methods (Kreyszig, 1970; Seinfeld and Lapidus, 1974). For these methods, all information is used for all streams, elements and nodes for which metallurgical accounting information is available. Repeating these methods using a Monte Carlo simulation approach based on known measurement uncertainty distributions allows one to generate the distribution in recovery estimates and therefore to calculate the recovery variance after reconciliation. Other possibilities are more elements assayed and better feed mass measurement and sampling. In essence this would aid the metallurgist to investigate various options to determine where the most variance reduction can be obtained for the least investment.
- It should be noted that no assumption of the distributions of measurement uncertainties were made. This does not influence analyses, except where standard errors are converted into standard deviations, or visa versa, where an assumption of normality is made. The normality assumption influences the 95% confidence intervals. A skewed (non-symmetrical) distribution will lead to non-symmetrical confidence limits around the expected value.
- The error propagation model is based on fundamental theorems in mathematics and statistics and material balances and is not constrained to a priori knowledge of the error distributions. However, practical implementation of the model requires a method to translate standard errors to standard deviations, and the quantification of the error distribution through fitting the histogram developed through well-planned sampling campaigns and/or the continuous use of well calibrated sampling and measurement devices. In short, we need a model of how measurement error itself behaves at different measurement values.
- Errors which lie outside the propagation of variance models presented above may be caused by bias or by data which is simply in error for some other reason such as mis-entry of data or mislabelling of samples.
- Large relative standard deviations or measurement variances may require a more complex variance propagation formulation, as the first term of the Taylor series approximation becomes inadequate to quantify the full effect of measurement errors on the variance in the calculated variable (recovery in this case).

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The AMIRA P754 programme is aimed at delivering a code, a set of guidelines and a set of toolboxes for metallurgical accounting and reconciliation with the aims of:

- improving transparency at the corporate governance level
- quantifying the causes and sources of 'unaccounted losses' and bias
- quantifying the uncertainty in measurements of masses, assays, stocks and recoveries
- quantifying the investment risk associated with uncertainties in working capital (inventories and stocks) and operating performance (recovery).

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