# Rough-wall and turbulent transition analyses for Bingham plastics

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## Synopsis

Some years ago, the authors developed an analysis of non-Newtonian pipe flow based on enhanced turbulent microscale and associated sublayer thickening. This analysis, which can be used for various rheologic modelling equations, has in the past been applied to smooth-walled pipes. The present paper extends the analysis to rough walls, using the popular Bingham plastic equation as a representative rheological behaviour in order to prepare friction-factor plots for comparison with measured results. The conditions for laminar-turbulent transition for both smooth and rough walls are also considered, and comparisons are made with earlier correlations and with experimental data.


## Introduction

In 1985 the present authors developed an analysis of non-Newtonian turbulent pipe flow based on enhanced turbulent microscale and associated sublayer thickening (Wilson and Thomas, 1985). This analysis, which can be used for various rheologic modelling equations, has in the past been applied to smooth-walled pipes. Recently, the analysis was used to develop an analytic model of laminar-turbulent transition for Bingham plastics (Wilson and Thomas, 2006).

The current paper extends the 1985 friction factor analysis to rough-wall pipes for the Bingham plastic rheologic model. Bingham plastic flow in rough pipes is of commercial relevance in many applications. For example, the hydraulic roughness of unlined long distance slurry pipelines generally increases with time and the pipelines require regular pigging. Steel tailings pipelines can similarly become significantly rougher with time. Pipe roughness is also relevant to low pressure concrete pipelines or flumes.

In addition to the rough-wall friction analysis, the rough-wall analysis is applied to the recent laminar-turbulent transition, resulting in an analytic model for laminar-turbulent transition in smooth and rough-wall pipes.

The results are compared with the very limited test data available for non-Newtonian flow in rough pipes.

## Previous smooth-wall analysis

For turbulent flow of a Newtonian fluid in a smooth-walled pipe the ratio of mean velocity VN to shear velocity $U^*$ is given by the well-known relation:

$$V_N/U^* = 2.5 \ln(\rho\, D\, U^*/\mu) \qquad [1]$$

where $U^*$ is shear velocity (equal to $\sqrt{(\tau_o/\rho)}$), $D$ is internal pipe diameter, $\rho$ is fluid density and $\mu$ is fluid viscosity. The ratio $\rho DU^*/\mu$ is called the shear Reynolds number, denoted $Re^*$.

For flow of a non-Newtonian fluid in a smooth-wall pipe, Wilson and Thomas (1985) presented a theory based on enhanced turbulent microscale and associated sublayer thickening. The sublayer thickness is increased by a factor $\alpha$ which equals the rate of the area underneath the non-Newtonian rheogram to the triangular area associated with an equivalent Newtonian fluid, with $\mu$ evaluated near the pipe wall where $\tau = \tau_o$.

For the commonly employed Bingham-plastic model, $\alpha$ depends only on the stress ratio $\theta$, defined as the ratio $\tau_o/\tau_B$, where $\tau_o$ is the shear stress at pipe wall and $\tau_B$ is the Bingham yield stress. At this point it is useful to introduce a dimensionless parameter developed by Hedström (1952)—the Hedström number $He$, defined as:

$$He = \rho\, \tau_B\, D^2/\, \eta_B^{\,2} \qquad [2]$$

As shown by Wilson and Thomas (2006), the mean velocity ratio for turbulent flow of a Bingham plastic in a smooth-walled pipe can then be written as:


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$$V/U^* = 2.5 \ln[(He)^{0.5}] + 2.5 \ln[\theta^{0.5} (\theta - 1)/(\theta + 1)] + 11.6/\theta - \Omega \qquad [3]$$

The final term in Equation [3] takes account of any blunting of the velocity profile near the pipe centre-line where $\tau < \tau_B$. As introduced by Wilson and Thomas (1985), for a Bingham plastic $\Omega$ depends only on $\theta$, and can be written:

$$\Omega = - 2.5 \ln[(\theta - 1)/\theta] - 2.5(\theta + 0.5)/\theta^2 \qquad [4]$$

$V/U^*$ can be written as $\sqrt{(8/f)}$ where $f$ is the Darcy-Weisbach friction factor. Thus $f$ for turbulent flow of a Bingham plastic in a smooth pipe can be determined directly from Equation [3] with Equation [4] inserted.

### Rough-wall friction analysis

The frictional effect of wall roughness, of representative height $k$, depends on the ratio between $k$ and the height $\delta$ of the viscous sublayer. If this ratio is small, the roughness is engulfed in the viscous sublayer, and friction is essentially the same as that for a smooth–walled pipe. As $k/\delta$ increases, friction increases initially, until $k$ exceeds $\delta$ and the rough-wall condition is achieved. The usual way of including this variation is by means of the Colebrook-White equation. For present purposes this equation can be expressed in the form that gives $V/U^*$ in terms of $Re^*$ and $k/D$, thus:

$$V/U^* = -2.5 \ln[(1/Re^* + (k/D)/3.7] \qquad [5]$$

Equation [5] can be rearranged to give:

$$V/U^* = 2.5 \ln(Re^*) + 2.5 \ln(3.7) - 2.5 \ln[3.7 + (k/D)Re^*] \qquad [6]$$

For the Newtonian fluid considered here, the first term on the right-hand side of Equation [6] corresponds to the smooth-wall value of $V/U^*$ (compare Equation [1]), and the remaining terms represent a correction for the effect of wall roughness. Since the effect of roughness is to reduce $V$, it is appropriate to use the smooth-wall method to get $V/U^*$ (for both Newtonian and non-Newtonian cases), and then subtract a rough-wall correction term $\Delta V/U^*$, given by:

$$\Delta V/U^* = - 3.27 + 2.5 \ln[3.7 + Re^*_k] \qquad [7]$$

Here 3.27 equals 2.5 ln(3.7) and, in the case of a Newtonian fluid, $Re^*_k$ represents $(k/D)Re^*$ or $\rho k U^*/\mu$.

It can be seen that $Re^*_k$ for a Newtonian fluid is proportional to the ratio of $k$ to sublayer thickness (which varies with $\mu/(\rho U^*)$). As noted earlier, in non-Newtonian flows the sublayer thickness is increased by a factor $\alpha$, and thus, for a non-Newtonian fluid

$$Re^*_k = \rho k U^*/(\alpha\, \mu) \qquad [8]$$

For the specific case of a Bingham plastic, which is of interest here, this becomes:

$$Re^*_k = [\rho k U^*/ \eta_B]\, (\theta - 1)/(\theta + 1) \qquad [9]$$

In summary, the Wilson-Thomas theory has been extended to rough-wall pipes carrying a turbulent flow of a Bingham plastic. First, $V/U^*$ is obtained for a smooth-wall pipe (Equations [3] and [4]) and then $\Delta V/U^*$ (Equations [7] and [9]) is subtracted to account for rough-wall effects. In the course of the calculations, parameters such as the Hedström number are first specified, and then a series of values of the shear ratio $\theta$ are inserted in the equations. Outputs include $V/U^*$ (and hence $f$) and the Bingham plastic Reynolds number ($Re_B = \rho VD/\eta_B$).

Figure 1 shows the modelled friction factor for two Bingham plastic fluids of $He$ = 1E5 and $He$ = 1E7. The friction factor is shown plotted against the Bingham-plastic Reynolds number $Re_B$. Predictions are shown for a smooth-wall pipe, and pipes with relative roughness ($k/D$), of 0.0003 and 0.001. Laminar-turbulent transition occurs at around $Re_B$ 7000 for $He$ = 1E5 and at around $Re_B$ 1E5 for $He$ = 1E7. Also shown as dashed curves are predictions for a Newtonian fluid for the same three relative roughness levels.

New steel pipe typically has a roughness of 0.05 mm so the $k/D$ = 0.0003 could apply to a steel pipe of ID 165 mm. A quite feasible increase in pipe roughness to 0.165 mm in the same size pipe would result in $k/D$ increasing to 0.001.

Figure 1 shows that roughness has a greater effect on the friction factor at the higher Hedström number. For $He$ = 1E7 at $Re_B$ = 3E5 the friction factor for $k/D$ = 0.001 is approximately 35% higher than in a smooth-walled pipe.

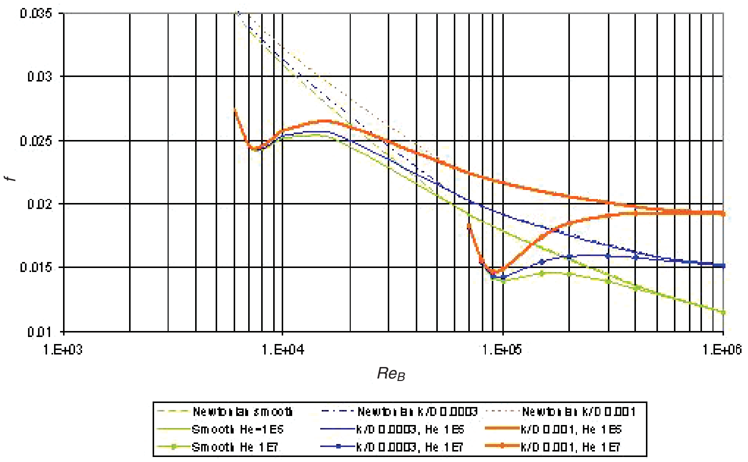


**Figure 1—Smooth and rough wall predictions for *He* = 1E5 and 1E7**

It is important to note that subtraction of $\Delta V/U^*$ from the smooth-wall equation is not simply the same adjustment as for a Newtonian fluid but allows for the increased engulfing of pipe roughness by the thicker sublayer.

### Comparing theory with data

Friction loss data for Bingham plastic flow in rough-wall pipelines is extremely limited. Some data have been obtained from operating long-distance slurry pipelines, but these data are generally available at only a specific flow rate. Wilson and Thomas (1985) did compare one set of data obtained in a 206 mm limestone pipeline with an approximate prediction obtained by increasing the smooth-wall friction by the same ratio as for a Newtonian fluid. Venton (1982) provides further general details of this 24-km pipeline. The present more rigorous rough-wall prediction method has been applied to these data and the results are shown in Figure 2. The predictions relate to a Bingham yield stress of 7.92 Pa and a plastic viscosity of 14.1 mPa s ($He$ = 2.77E6). The relative pipe roughness $k/D$ = 0.00079.

The data are seen to lie somewhat above the turbulent flow prediction but below the Newtonian rough case, and the trend of the data is in accord with the predicted trend.

### Analysis of data of Slatter and Van Sittert

To the authors' knowledge the only test data specifically aimed at investigating the effect of pipe roughness on non-Newtonian slurry flow are that of Slatter and Van Sittert (1997 and 1999). These authors tested kaolin and platinum tailings slurries in 45.3 mm and 27.1 mm pipes of varying roughness ranging from 47 microns to 693 microns ($k/D$ ranging from 0.0010 to 0.0153). All slurries were modelled as yield power law although in one case (platinum tailings SG 1.801) the yield power law exponent is 1.00128, meaning that a Bingham model is essentially appropriate.

At this point it should be noted that Slatter and Van Sittert (1999) compared their data with three prediction methods, Torrance (1963), Wilson and Thomas (1985) and Slatter (1995). It is unclear how Slatter and Van Sittert adapted the smooth-wall Wilson-Thomas method to rough-wall prediction. However, the current authors believe it unlikely that full allowance for the effect of the thickened sublayer engulfing pipe roughness was undertaken. In general, Slatter and Van Sittert found all three methods overpredicted for the platinum tailings flows. Space limitations preclude presenting detailed comparisons here, but it is worth noting that their observed friction factors for turbulent flow lay between the smooth-wall and rough-wall predictions obtained from the present model.

In the case of the laminar flow of their kaolin slurries, Slatter and Van Sittert chose to use a three-parameter yield-power-law rheological model. Three-parameter models offer an apparent increase in fitting accuracy, but at the cost of uncertainty in making adequate estimates of so many parameters. This can be especially worrisome for rough pipes where $k/D$ constitutes yet another parameter. Thus, it is interesting to re-analyse these data in terms of the two-parameter Bingham plastic model. Figure 3 shows laminar flow curves (shear stress versus shear rate) for the kaolin SG 1.0803. The thick grey curve represents the flow curve for a yield power fluid with yield stress 5.505 Pa, $K$ = 0.28785 and $n$ = 0.4626 as determined by Slatter and Van Sittert (1999). The laminar flow data were obtained at apparent shear rates up to about 1 300 s$^{-1}$. Maximum shear rates vary with type of equipment. Using rotational viscometers suitable for slurry testing, the maximum shear rate is usually in the range from about 500 s$^{-1}$ to about 1500 s$^{-1}$.

Shown in Figure 3 are three Bingham approximations to the generated yield-power-law curve. One Bingham line is based on the slope at 500 s$^{-1}$ ($\tau_y$ = 8.5 Pa, $\eta_B$ = 4.4 mPa s), one at 1 000 s$^{-1}$ ($\tau_y$ = 9.4 Pa, $\eta_B$ = 3.2 mPa s), and one at 1 500 s$^{-1}$ ($\tau_y$ = 10.1 Pa, $\eta_B$ = 2.6 mPa s).

Figure 4 shows the predicted friction factor in the roughest 45.3 mm pipe ($k/D$ = 0.0153) based on the Bingham yield stress 8.5 Pa approximation ($He$ = 9.73 E5). Also shown in Figure 4 is the trend of the experimental data, scaled from the Slatter and Van Sittert graphs and represented by a few representative points.

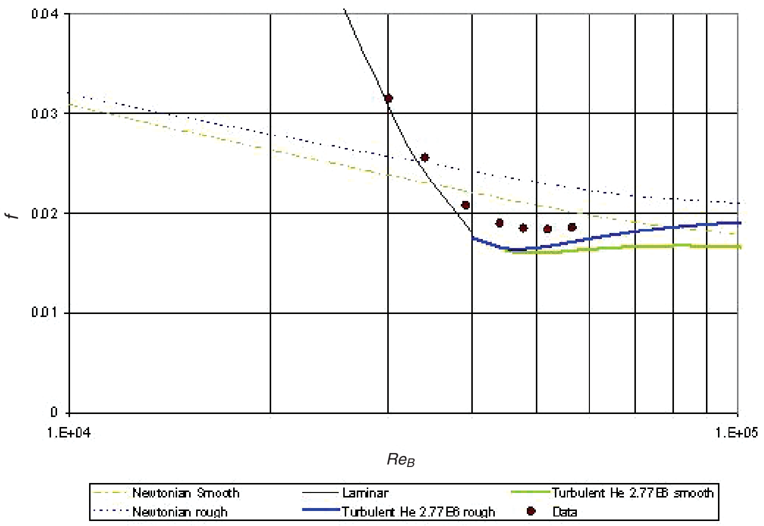


**Figure 2—Rough-wall prediction compared with limestone data**

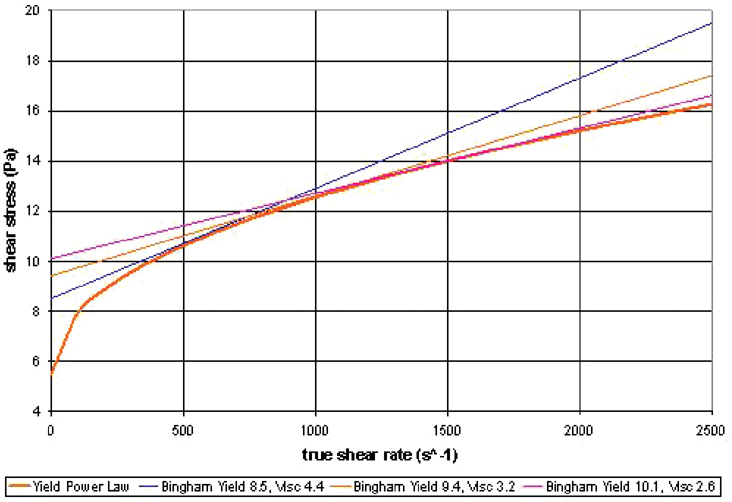


**Figure 3—Bingham approxomations to yield power law flow curve kaolin SG 1.0803**

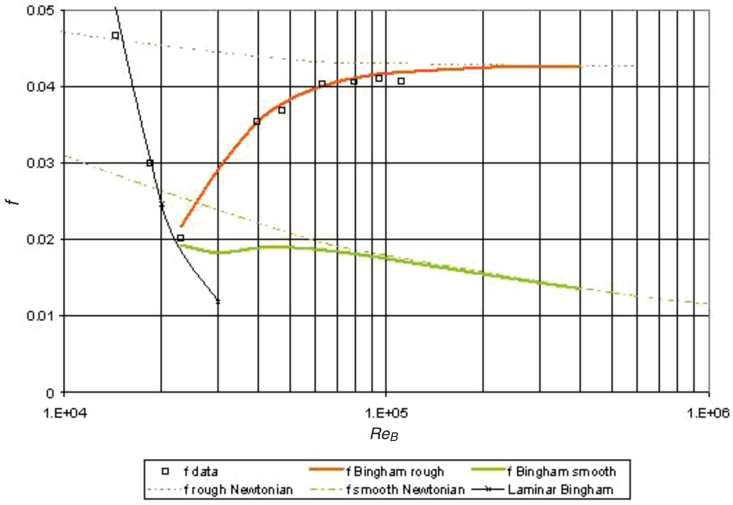


**Figure 4—Kaolin rough-wall prediction—Bingham yield stress 8.5 Pa**

Figure 4 indicates excellent agreement between prediction and the observed data trend. There is a predicted and observed halving of the friction factor between $Re_B$ = 1 E5 and transition at $Re_B$ = 2.3 E4. This reduction in friction factor is explained by the thickening of the viscous sublayer with resulting engulfing of the roughness within the sublayer. Without allowance for this effect the friction factor would follow the dotted rough-wall Newtonian curve.

Figure 5 shows a similar comparison for the same data with predictions based on the Bingham approximation with yield stress 9.4 Pa (*He* = 2.03 E6). Once again the agreement is excellent.

Finally Figure 6 shows a similar comparison with predictions based on the Bingham approximation with yield stress 10.1 Pa (*He* = 3.31 E6). The agreement is excellent.

Thus Figures 4, 5 and 6 indicate similar predictions obtained using Bingham plastic approximations based on tangents to the yield power law flow curve at 500, 1 000 and 1 500 $s^{-1}$ shear rates. This is an important finding since use of the Bingham plastic model avoids a fundamental shortcoming of the yield power law model.

As noted by Slatter and Van Sittert, ideally laminar flow data should be obtained at similar shear stress values as will apply in turbulent flow. However, this is generally not possible and so a rheological model is fitted to the laminar flow data and is assumed capable of extrapolation to the shear stress levels applying in turbulent flow. The problem with the yield power law model is that at the high shear stress applying in turbulent flow the apparent (secant) viscosity approaches zero.

For example, in the case of the kaolin data in Figures 4, 5 and 6 the highest wall shear stress in turbulent flow is 550 Pa. Extrapolation of the yield power law model ($\tau_y$ = 5.505 Pa, $K$ = 0.2878, $n$ = 0.4626) to a shear stress of 550 Pa requires a laminar flow shear rate of 1.2 E7 $s^{-1}$. At this shear rate the apparent (secant) viscosity is only 0.0456 mPa s, 1/20th of the viscosity of water. This is obviously physically unrealistic. In contrast to the yield power law model, the apparent viscosity of the Bingham plastic model approaches the plastic viscosity at very high shear rates, a much more physically realistic scenario.

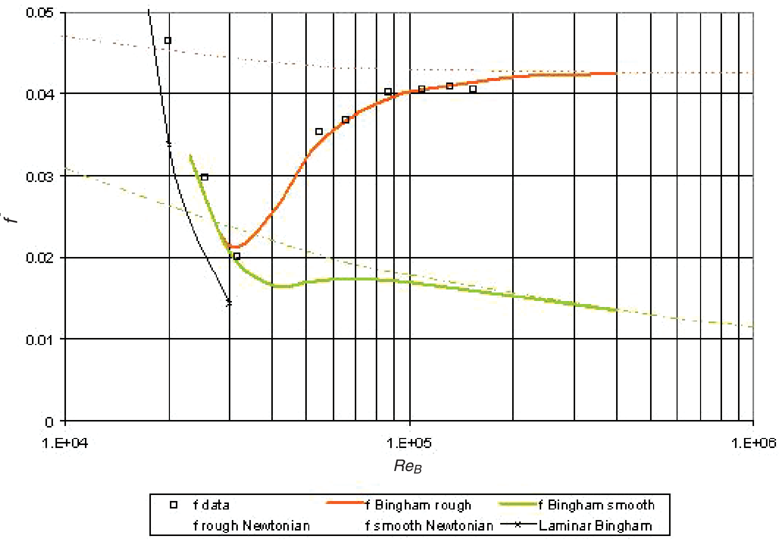


**Figure 5—Kaolin rough-wall prediction—Bingham yield stress 9.4 Pa**

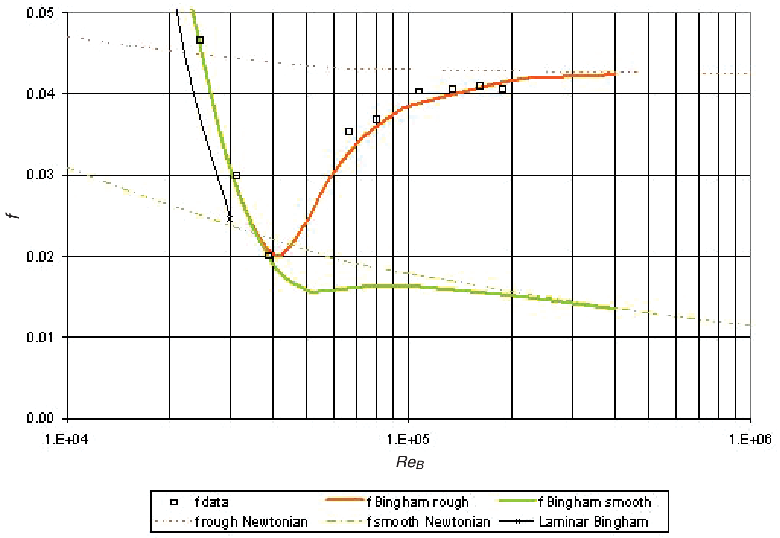


**Figure 6—Kaolin rough-wall prediction—Bingham yield stress 10.1 Pa**

Although much more work is required to fit to flow models to laminar flow curves and consequences for extrapolation to higher shear rates, the limited comparisons in Figures 4, 5 and 6 suggest that the Bingham model fitted to the highest shear rate data available allows suitable predictions and is preferred to the yield power law model.

### Influences of roughness on transition velocity

A close inspection of the predictions for $He$ = 1 E7 in Figure 1 indicates a slight reduction in the laminar-turbulent transition velocity as the roughness increases. This reduction in transition velocity with increased roughness has been analysed in detail and the results shown in Figure 7 as $V_{trel}$ versus Hedström number for various relative roughness where $V_{trel}$ is given by:

$$V_{trel} = V_t / \sqrt{(\tau_y/\rho)} \qquad [10]$$

The Wilson and Thomas (2006) analysis for smooth-wall pipes showed that, for $He$ > 1E5, $V_{trel}$ has the value 25.

Figure 7 indicates that roughness significantly influences the transition velocity only when the relative roughness exceeds about 0.001. The effect is approximated by the equation:

$$V_{trel}(rough)/V_{trel}(smooth) = e^{-2.5k/D} \qquad [11]$$

### Conclusion

Some years ago, the authors developed an analysis of non-Newtonian pipe flow based on enhanced turbulent microscale and associated sublayer thickening. In the past, this analysis was applied to smooth-walled pipes. The present paper extends the analysis to rough walls, using the popular Bingham plastic rheologic model. The conditions for laminar-turbulent transition for both smooth and rough walls are also considered.

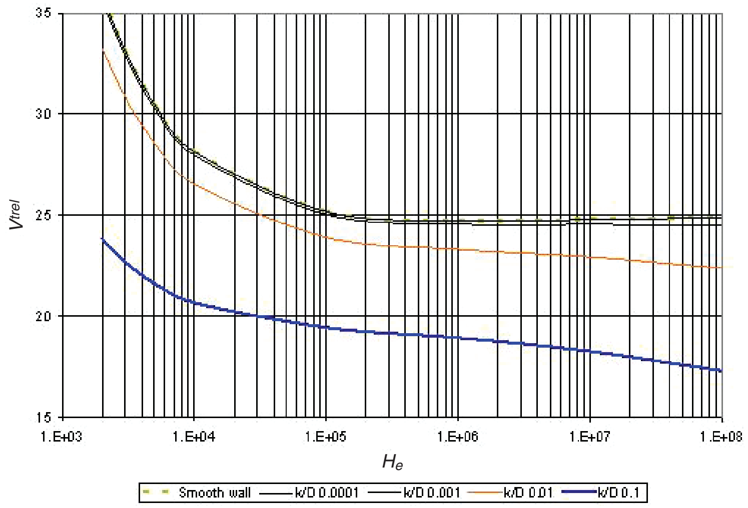


**Figure 7—Effect of roughness on transition velocity**

The frictional effect of wall roughness, of representative height $k$, depends on the ratio between $k$ and the height $\delta$ of the viscous sublayer. If this ratio is small, the roughness is engulfed in the viscous sublayer, and friction is essentially the same as that for a smooth–walled pipe. As $k/\delta$ increases, friction increases initially, until $k$ exceeds $\delta$ and the rough-wall condition is achieved. Equations for friction factor are obtained using the Colebrook-White equation and the expressions derived previously for viscous sublayer height.

These equations are used to prepare friction-factor graphs, which are compared with the rough-wall data obtained by Slatter and Van Sittart. As shown in Figures 4, 5 and 6, the predictions show very satisfactory agreement with the data.

## Notation

| | |
|---|---|
| $D$ | internal pipe diameter |
| $f$ | friction factor (Darcy-Weisbach) |
| $f_T$ | value of $f$ at laminar-turbulent transition |
| $He$ | Hedström number (see Equation [2]) |
| $k$ | representative roughness height |
| $Re^*$ | shear Reynolds number ($\rho DU^*/\mu$) |
| $Re^*_{\mathrm{k}}$ | Reynolds number based on roughness ($\rho kU^*/\mu$) |
| $Re_B$ | Bingham plastic Reynolds number ($\rho VD/\eta_B$) |
| $U$ | local velocity at distance $y$ from wall |
| $U^*$ | shear velocity ($\sqrt{[\tau_o/\rho]}$) |
| $V$ | mean (flow) velocity |
| $V_N$ | mean (flow) velocity for a Newtonian fluid |
| $V_t$ | value of $V$ at laminar-turbulent transition |
| $V_{trel}$ | relative transition velocity $V_t/(\tau_B/\rho)^{0.5}$ |
| $y$ | distance from pipe wall |
| $\alpha$ | area ratio of rheogram |
| $\delta$ | thickness of viscous sublayer |
| $\eta_B$ | Bingham (tangent) viscosity |
| $\theta$ | shear stress ratio $\tau_o/\tau_B$ |
| $\mu$ | secant viscosity |
| $\rho$ | density of material |
| $\tau$ | shear stress |
| $\tau_B$ | Bingham (yield) shear stress |
| $\tau_o$ | shear stress at pipe wall |
| $\Omega$ | effect of $\tau_B$ on velocity-profile blunting |

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